TEACHING MATHEMATICS

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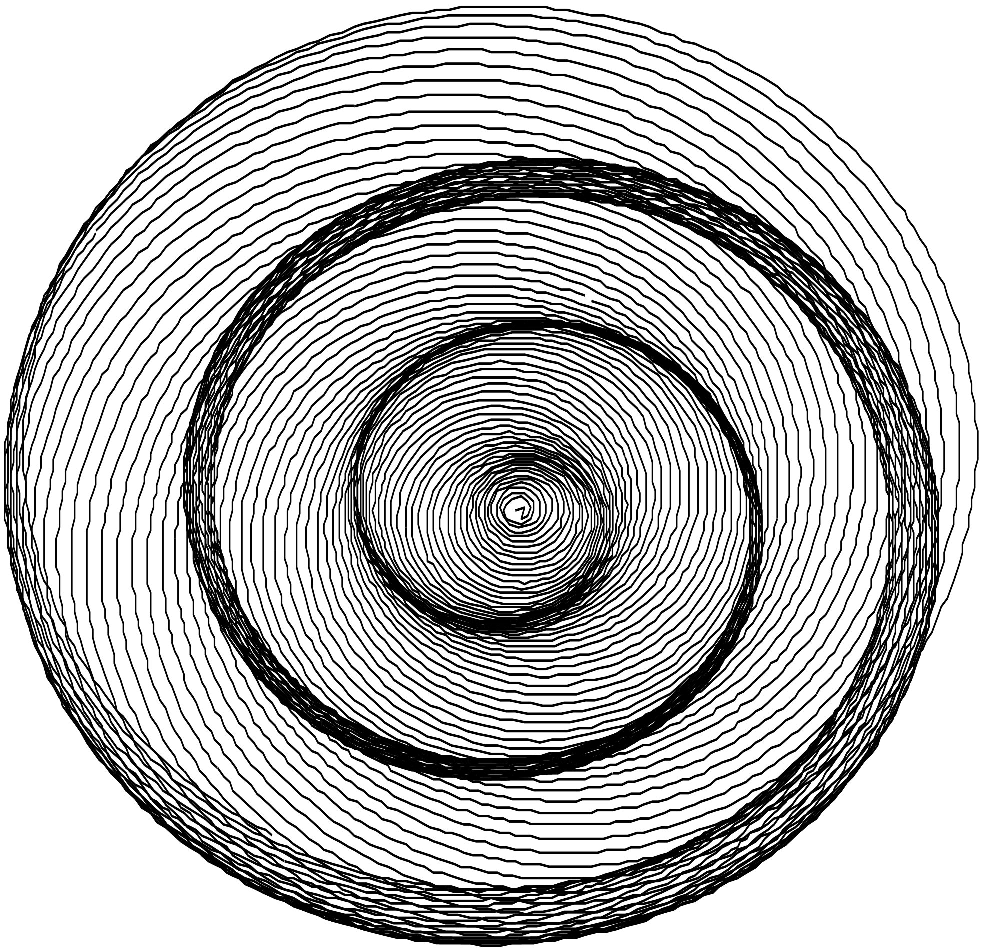
The Journal of the Queensland Association of Mathematics Teachers Inc



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Teaching Mathematics is the official journal of the Queensland Association of Mathematics Teachers Inc. The journal is published four times a year (March, June, August and November) and is provided as a right to all members of the association.

The aims of the journal reflect, put into effect, and communicate the objectives of the association:

- to promote interest in mathematical education at all levels;
- to encourage and promote research in the teaching of mathematics;
- and to speak and act on matters related to mathematical education.

Educators, administrators, parents and students are invited and encouraged to contribute articles which relate to the teaching of mathematics at any level. These may be of any length from interesting 2-line snippets, through short letters concerned with topical issues to longer in-depth articles. Critical comments and advice about the future directions of the journal are always welcome.

Materials should be sent to the editor, David Ilsley. The preferred way is by e-mail to ilsley@cyberbiz.net.au Materials may also be sent on floppy disk. Contact details are as follows:

David Ilsley	Phone (home):	(07) 3807 1327
Queensland School Curriculum Council	Phone (work):	(07) 3228 1501
PO Box 317	Fax (work):	(07) 3228 1555
Brisbane Albert Street	e-mail (home):	ilsley@cyberbiz.net.au
Qld 4002	e-mail (work):	david.ilsley@qscc.qld.edu.au

Word for IBM (up to Word 97) is the preferred format. All receipts will be acknowledged – if you haven't heard within a week, ring David to check.

Copy dates are: mid-February; mid-May; mid-July; mid-October.

If you have any questions, contact David Ilsley.

Publications sub-committee members are listed below. Feel free to contact any of these concerning other publication matters.

Books, software etc. for review should be sent to Lydia Commins; information to go in the newsletter should be sent to Sue Reid.

Newsletter copy dates are a month earlier than those for the journal.

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QAMT Office:

QAMT Inc ***Phone and fax : (07) 3864 3920***
PO Box 328 e-mail: qamti@gil.com.au
Everton Park 4053

Office hours: 9 – 4, Wednesday and Thursday, term weeks and some holiday weeks
Office Administrator: June Nowitzki

Membership and advertising enquiries: contact the QAMT office.

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TEACHING MATHEMATICS

CONTENTS

Regular Features

Conference Proceedings

Special Features

FROM THE PRESIDENT

John McKinlay
Carmel College

Welcome to another year of QAMT activities – the last year of the millennium (depending on your point of view).

QAMT is expanding its list of publications. The journal, the newsletter and Tenrag have been published now for many years. 1997 saw the publication of the AToMIC Project and in 1999 work is continuing on the SubAToMIC Project (Subsequent Applications to Mathematics Incorporating Calculators) for lower secondary students. In addition, a book of Application assessment items for Maths A, B and C is being considered. It is hoped that QAMT will continue to produce valuable publications throughout the years to come.

Of course the big event coming up is the May Day Conference at the Brisbane Novotel. The May Day conferences have been held at the Bernard O'Shea Centre for some years now, but it was finally decided that we have outgrown the venue. We look forward to the new experience. Maree Mortlock is busy putting the final touches on the program and it looks like it will be an exciting and rewarding day.

The annual conference is of course in Rockhampton this year – our first venture outside of the south-east Queensland. Kiddy, Rex, Robert and others have been working hard securing a range of accommodation at very reasonable (even cheap) prices and a special $36 return fare from Brisbane on the tilt train. Bring the family for a bit of a getaway. There will be a three day conference program (Wednesday 29 September to Friday 1 October) and plenty of things for the family to do should they not want to attend the presentations.

Please note that I have moved from Windaroo SHS to QSCC. I now have an e-mail address at home and this is the best way to send contributions and articles for the journal. All the details are on the inside front cover.

FROM THE EDITOR

David Ilsley
Queensland School Curriculum Council

Welcome to another issue of *Teaching Mathematics*. You may have noticed that the size of the journal is gradually increasing. This is a reflection of the amount of material being received for publication. It is great to get written versions of conference presentations. This allows the many readers of the journal to benefit from the ideas presented, even if they could not be there in person to get them first hand. As well as the conference proceedings, there is a steady supply of other articles. The efforts of all those who have contributed are gratefully acknowledged. Their work is what makes this journal a valuable contribution to the professional growth of teachers around the state, and also helps to make the QAMT an organisation worth belonging to. It would be great to see this growth continue into the future.

Another service provided by the QAMT is our website. We are not at present taking full advantage of this opportunity to assist teachers with their work. Much of the information on the QAMT site is now out of date and other material may not be relevant. Peter Cooper is currently planning a major improvement of the site and funds have been set aside to facilitate this.

Peter is looking for suggestions as to what the site might contain which you would find either useful or interesting. For example, QAMTAC'99 is intending to use the web-site to publish programs, abstracts and various resources that can be downloaded. If you have any other ideas, please contact Peter at spcooper@uq.net.au . Visit http://qamt.cqu.edu.au to see what you think might be worth retaining or developing (and if you forget the address it is published at the bottom of the back inside cover of each issue of the journal).

I recently tendered my resignation as Publications Convenor and as Vice President of the association. This was not through a lack of desire to do the jobs, but because of a work commitment which prevents me from attending executive meetings. Brad Barker is now filling these roles and Merrilyn Goos had joined the executive as Assistant Secretary. I will continue to edit the journal.

What's happening?

in mathematics education

This will be a semi-regular feature of the journal. It is designed to help readers keep up with developments in mathematics education in Queensland and elsewhere. This item presents an overview of the National Literacy and Numeracy Plan and the Supporting Literacy and Numeracy in Queensland Schools Project.

The National Literacy and Numeracy Plan
The Supporting Literacy and Numeracy in Queensland Schools Project

In 1996 to 1997 the Ministerial Council for Education, Employment, Training and Youth Affairs (MCEETYA) established the National Literacy and Numeracy Plan. This plan focuses on the early years of schooling and emphasises a range of actions by states and territories including:

- early assessment and identification of students at risk
- early intervention
- regular assessment against agreed national benchmarks
- national reporting of student achievement, and
- professional development for teachers in improving literacy and numeracy learning outcomes.

The National Literacy and Numeracy Benchmarks are the main initiative to come out of this plan.

Queensland launched the Supporting Literacy and Numeracy In Queensland Schools Project to support implementation of the national plan. This project is a Commonwealth-funded joint initiative of Education Queensland, the Queensland Catholic Education Commission and the Association of Independent Schools of Queensland.. The project is being coordinated by Education Queensland's Centre for Teaching Excellence

The Queensland project will run through 1998 and 1999 and will result two main products in the numeracy area – Support a Maths Learner: Number and Developmental Continuum and Intervention Strategies for Space, Measurement and Data.

Support a Maths Learner: Number

This is a package of training workshops for use by program coordinators, and intervention resources for use by trained teacher aides, parents and volunteers to help children experiencing difficulties in Number.

This resource was distributed to state, Catholic and independent schools across Queensland in term 4 1998. State schools were required to request their complimentary copy of the resource. Schools can still obtain their complimentary copy by contacting CTE. Additional copies can also be purchased at a cost of $55. Order forms for additional copies are included in each kit.

Developmental Continuum and Intervention Strategies for Space, Measurement and Data

This resource complements the Continuum and Intervention Strategies for Number. It is a teacher resource to facilitate mapping children's learning and development, identifying children experiencing difficulties, supporting children's learning and interpreting relevant strands of the National Numeracy Benchmarks for Years 3 and 5.

The continuum and intervention strategies as well as a multimedia professional development resource will be distributed later in 1999.

1999 Mayday

From left to right: Sue Muir, Dympna Ridby, Karen Jackson and Lyn Northdurft getting ready for a session on Mathematics through Fairytales by Merrilyn Goos.

About 160 delegates attended this conference with about half this number being primary teachers. What a great roll up! There were plenty of sessions to go to, a range of interesting topics being discussed and many trade displays to visit.

Right: Potential Cinderella (Matine Suzor) being measured up by Prince Charming (John McKinlay) to see if the shoe fits in the Fairytale Mathematics session. Who is that injured man in the background?

Conference

Our guest speaker was Judy Moseley who spoke to us about making sense of mathematics for our children. Judy works at Deakin University and in conjunction with Peter Sullivan and Peter Mousley has recently produced a CD entitled “Learning About Teaching”.

Right: Maree Mortlock presenting keynote speaker Judy Moseley with some beautiful wine glasses.

From left to right: Peter Fas, Louise Nott, Lew Johnson and Dianne Jensen enjoying morning tea.

From the Journals

Merrilyn Goos
The University of Queensland

The Australian Mathematics Teacher, Volume 54, Number 4 (March 1999).

The lead article in this issue, by Barry Kissane and Marian Kemp, considers the appropriateness of using graphics calculators in lower secondary mathematics classrooms. While graphics calculators are becoming increasingly popular in the upper secondary school, their use in junior mathematics classes raises several issues which are addressed in this article. Arguing in favour of integrating this kind of technology into lower secondary mathematics teaching, Kissane and Kemp point out that graphics calculators provide significant opportunities to improve students' learning by allowing them to analyse and explore mathematical ideas in new and powerful ways. In addition, the availability of low cost entry level calculators now makes it feasible for junior students to own a graphics calculator instead of the more common, but less versatile, scientific calculator. The authors describe several specific advantages of graphics calculators over scientific calculators for learning lower secondary mathematics, such as superior screen capabilities, use of conventional algebraic syntax, the ability to connect graphical, symbolic, and numeric representations of functions, and statistical capabilities that allow for data storage and editing as well as different types of graphical summaries.

Despite the demonstrated relevance of graphics calculators to lower secondary students, however, some thorny issues remain unresolved. The most important concern is the likely impact of technology on curriculum content and assessment. In both cases, the authors propose that technology needs to be integrated rather than treated as an optional extra, and that reforms in these areas will require that all students have continuous personal access to appropriate technology. Kissane and Kemp contend that the affordability of low-end graphics calculators (currently around the cost of two CDs) makes such a situation possible.

This issue of *AMT* also contains an interesting report on the recent AAMT Virtual Conference. The report weighs up the pros and cons of web-based conferences against face to face gatherings, and touches on the difficulties of participating in Internet chats, down-loading software needed to unpack conference files, and keeping up with the pace of e-mail discussion.

Micro Math, Volume 15, Number 1 (Spring 1999).

Micro Math is a British journal, published by the Association of Teachers of Mathematics (web address http://mcs.open.ac.uk/cme/micromath/), and containing articles which offer practical ideas for teaching mathematics with technology. The current issue includes a report on the Secondary School sub-group of Working Group 16 at the International Congress on Mathematical Education (ICME8), held in Seville in 1996. The focus for this group was on the role of technology in the mathematics classroom. The authors (Hudson and Borba) summarise papers on a variety of topics from the UK, Malaysia, Japan, South Africa, Brazil, and France. Interestingly, similar themes and interests were evident in contributions from this wide range of countries and cultures: the role of technology in improving learning; the changing role of the teacher; increasing emphasis on mathematical processes (such as conjecturing and testing, inquiry and investigation) and decreasing emphasis on calculation and manipulating symbols; curriculum issues (changes to sequencing, how technology makes some topics obsolete while making others more accessible); and the need for changes in assessment practices.

More practically oriented articles are contributed by Alison Clark-Jeavons, who describes how she taught her Year 8 students to program their TI-82s to produce animated pictures *(TI-magotchi)*, and Daniel Scher, who shows how dynamic geometry software can become a partner in problem solving and proof *(Problem solving and proof in the age of dynamic geometry)*.

Teaching Mathematics for Understanding

Some Reflections

Judith A. Mousley
Deakin University

This paper is a summary of the ideas presented in a plenary paper to the May 1999 conference of the Queensland Association of Mathematics Teachers. It uses reflections on past incidents to discuss layers of understanding that need to be developed in maths classrooms. The conference presentation also focused on predictions, and a paper on possible future characteristics of mathematics education will appear in a later issue of the QAMT journal.

Reflections

The focus of the QAMT conference on looking back and looking forward in mathematics education gave me an opportunity to reflect on my years as a student teacher, as a primary then pre-school and secondary teacher, and later as a tertiary lecturer and researcher in mathematics education. I decided to draw out of that experience some ideas about what we currently ask of our students, what we should ask of them, and where we need to set our sites in mathematics education.

I am going to use examples not from my institutional experience, but from my intimate knowledge of one learner—one of our sons who took a particular interest in maths from a very early age. I do this because as teachers we learn a lot about our teaching, but we rarely have the opportunity to “get into the heads” of children, and we do not often hear them talking spontaneously about their mathematical ideas and experiences. Peter is the only learner of mathematics that I have been able to watch develop over 25 years and the only one whom I could question regularly as his maths concepts, language and skills developed—he loved playing with and talking about maths then, and still does.

However, he was not an exceptional child. Whenever I tell such stories, many parents and teachers come up later and tell me about similar events, reporting on insights that their children have had. Some contact me after asking their own children the same sorts of questions and being excited by the responses. Again, the children’s understandings are usually not gleaned in the hurly-burly of classrooms but during quiet conversations and games amongst family members or friendly, ad hoc interactions in schools. Many teachers and parents tell stories of children becoming entranced with difficult ideas like infinity, negative numbers and square numbers in their pre-school and early primary years. I find children’s enchantment with mathematics fascinating.

What’s the next number?

We had been travelling for a year, and were north of Mt Isa on the way to the Gulf. Peter was nearly five, and for his pre-school education we used the excellent correspondence materials available for Queensland’s children living in remote areas. When the novelty of games and songs in the car wore off, we used to fill the time with maths activities that included number play like “How many wheels will the next truck have?” and “What’s the next number?”

Mum: What’s after six?

Peter: Seven. What’s after one hundred?

Dad: A hundred and one. What’s after a billion?

Peter: A billion and one. (Pause) What’s the biggest number?

Dad: There is none.

Peter: (Long silence)

Mum: What’s wrong?

Peter: I can’t work it out.

Mum: That’s right. Nobody can. You can always say the next number.

Peter: Like one thousand hundred billion million and one.

Mum: Yes.

Peter: (Long silence)

Dad: Don't worry. You don't have to understand.

Peter: I do understand. I know it. It is beautiful. I love it.*

*These dialogues are reconstructed. Sometimes I remember who said what and exact words, and I always remember the meanings expressed by children. However, I do not remember exact examples, and have created some to suit.

Perhaps that is the day that Peter fell in love with mathematics—when he realised that it was so great and so abstract that he could not know it.

What levels of understanding do we demand of our students?

In schools, students are expected to learn about *objects*–counters, graphs, models, shapes, and even more abstract objects like variables, theorems and formulae. They are given a rich variety of everyday experiences in coming to understand these objects. They also learn to manipulate the objects appropriately. There are traditions to be followed here, such as grouping, adding, increasing according to a set ratio, rotating, and so on. Thus object-based understandings form a coherent network of concepts relating to the world of objects and the way we operate with them in mathematics.

From these experiences with objects, students develop an understanding of matching *symbol*. These include symbols that we name and describe objects with, such as the symbols for "two", "limit", or "theta". They also include symbols used to name and describe actions, such as symbols for "divided by" or "square root of". There are also traditional symbols that represent ideas, such as symbols representing equality, the cartesian areas of multiplication and the waves of trigonometric graphs. Thus children seek another level of understanding—a coherent knowledge network of symbols themselves as well as symbols-based operations. At first the symbols are bound to objects and actions on these, but gradually they become an entity in their own right and thus can be used as foundation objects for further operations and for more complex symbolisation.

These levels are somewhat sequential within concepts, but not within time—they apply to the learning of each new concept throughout schooling. Objects are not just concrete manipulatives, and not the province of primary schools. For instance, children might first learn subtraction with base ten materials such as bundles of straws or MAB blocks. To learn factorisation, junior secondary students might also use an object as they draw, partition and rename a rectangle sized $x + 3$ by $x + 2$ as $x^2 + 5x + 6$. Senior students might learn about integral notation using drawings of curves and some rectangles that fit under them. Any of these students might learn about bank interest through their every day, objective (and also subjective) experience.

Figure 1 (adapted from Schoenfeld, 1986) represents the types of understanding and relationships between them that have been described to date. The physical and symbolic components each need internal coherence, and direct links need to be made between these two realms.

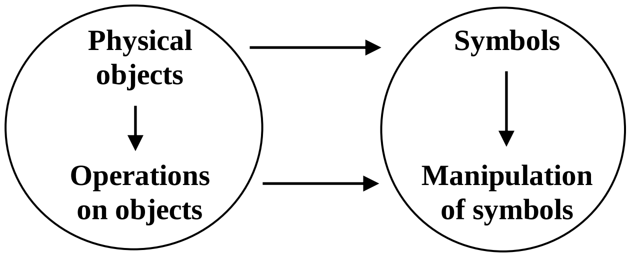


Figure 1: Object-based and symbol-based realms of understanding

A further level of understanding involves *abstraction* of symbol-based activity away from objects. Many teachers make good links between concrete experiences and abstract ideas, but don't realise the importance of taking the next step—abstracting mathematics not just out of, but away from, particular objective experiences.

Unfortunately, the symbolic and abstracted realms have got a bad reputation, but this is because they have been used without a strong foundation in the object realm. Developed properly, they are absolutely essential and useful components of mathematical understanding. The challenge here is for children to be able to recall enough relevant experience for the abstract to remain experientially meaningful, but to make the children's understanding independent of particular experiences. Hence I use the term "abstracted" rather than "abstract". It is in the fact that mathematics is removed from particulars that makes it powerful.

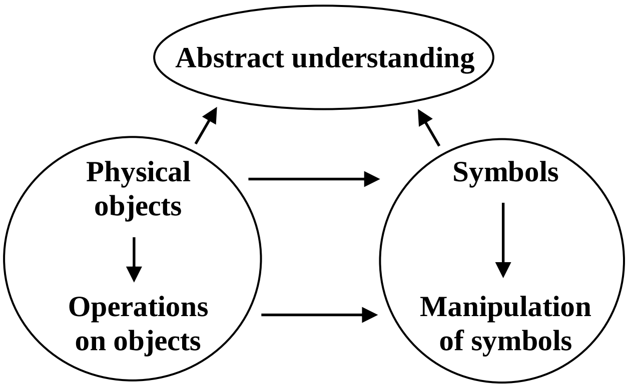


Figure 2: Object-based, symbol-based, and abstracted realms of understanding

It is rare for teachers to encourage students to reach a further level of abstraction—knowledge of *mathematics as an interlocking network of ideas*. We do it occasionally, such as when we explain multiplication as repeated addition, but this is usually within components of the curriculum, such as within the Number area. Reaching for this level of understanding (what Skemp, 1986 calls "relational understanding") involves using activities to help students to reflect on inter-relationships between concepts, linking them to create a network of understanding that involves personal meaning and interest.

And then comes an even higher level of understanding (see Figure 3) that involves students coming to know about *themselves as knowers of mathematics*. Again, this is not something that only senior students experience—as Peter's recognition of his own appreciation of the concept of infinity demonstrates.

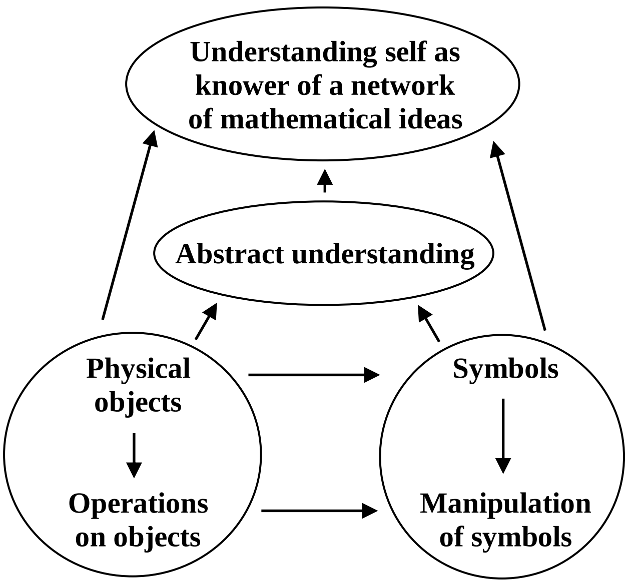


Figure 3: Self knowledge: A higher realm of understanding

What is sense making?

Peter was in a Year 2/3/4 composite class, but officially a Year 2 pupil. The teacher let children who had finished their work attempt any other work of their choice, so he often told me what the Year 3 and 4 children were learning.

Peter: I can do subtraction.

Mum: You've done that for ages—it's take away.

Peter: No, subtraction. Like 423 subtract 265.

Mum: Same thing … and how would you do that?

Peter: (paused for a few seconds) It's 100 ... 50 ... 8 One hundred and fifty eight.

Mum: (I wrote the algorithm and checked. He was right. I gave him more examples. Each one was calculated mentally and quickly.) How do you work them out? What are you thinking? Show me with this one.

Peter:

```
  4 2 3
– 2 6 5
```

Two hundred (pointing to the hundreds), backwards forty (pointing to the tens), backwards two (pointing to the units).

```
    1 1
  4 2 3
– 2 6 5
  1 5 8
```

Then he filled in the answer (1, then 5, then 8), crossed out some numbers, and wrote in little ones.

Mum: (after trying a few more examples with pencil and paper and wondering why I had never come across this method) You don't need to cross out numbers with your method. Why did you do that?

Peter: To get it right. It gets marked wrong if you don't.

Mum: (amused) What's this crossing out for? What does it mean?

Peter: When you don't have enough here (pointing to the 3) you can take a ten from here (the 2). The little one is the ten that you move across to the ones column. And then you need to take a hundred from here (the 4).

Mum: So then you can take it away. That's good. Why don't you do it that way?

Peter: Because it's stupid. It doesn't make sense.

The procedure certainly appeared to make sense to Peter. He was able to explain it the way the teacher would have, and his knowledge of place value was strong enough to support understanding of this procedure. Making sense to an individual, though, involves that person's willingness to engage personally with the procedures and its underlying ideas.

In refusing to engage, Peter was demonstrating a further level of understanding that involves sense of self as a knower and strategic user of mathematics. It is clear that he had decided to use the method he felt most comfortable with, and drew on his understanding of an idea (directed number) that is the essence of subtraction. In doing this, he had confidence in himself as a knower of the mathematics involved. His setting out of written work as the teacher expected, in order to "get it right" demonstrated a willingness to meet the norms of institutionalised schooling—not a form of mathematical understanding but a form of understanding related to happy existence in mathematics classrooms.

Since this time, and particularly when analysing videotapes of children working with a variety of procedures, I have seen the same pattern of action. I have had also heard and read of many such incidents that demonstrate all four realms of mathematical understanding.

How can we focus on underlying meaning?

Teachers allowing and encouraging the development of abstracted, related understandings, as well as the development of knowledge of oneself as a knower of mathematics, involve opening up the curriculum in strategic ways.

One way is to start with children's ideas, and then permit enough freedom in the curriculum to allow the big ideas children have to surface as they explore the objective and symbolic world. I have written many papers on ways that teachers have done this. For instance, Mousley (1990a) tells how a Year 1 teacher set up a problem-posing environment where children's varied interests in autumn leaves and other objects led to an exploration of ratio. In Mousley (1990b), I report on how a Year 6 teacher developed a lesson out of a child's asking why Toblerone bars are shaped as they are. Mousley (1990c) reports on the learning resulting from a Year 12 assessment task where students explored the notion of fractals.

Another strategy for emphasising thinking is to ask more open questions. Consider the following examples (taken from Sullivan & Mousley, 1999) and ask what they have in common.

- The mean height of four people in this room is 155 cm. You are one of those people. Who are the other three?
- A ladder reaches 10 metres up a wall. How long might be the ladder, and what angle might it make with the wall?
- What are some functions that have a turning point at (1,2)?
- At what times do the hands of a clock make an angle of 90°?
- A rectangle has a perimeter of 64 cm. What might its area be?

Open-ended tasks engage students in constructive thinking by requiring them to consider the broader possibilities, to seek patterns, to generalise. They are not abstract problems, but they encourage abstraction.

Try one of the above examples, and think about yourself as a learner: How did you start? What concepts did you use and what skills did you practise that were at first not obvious in the task? Where did you get to, and what might be an end point?

Finding multiple solutions for such problems should not be seen as an end in itself. There's potential to extend every one of these problems. For instance, with older children and the rectangle problem, they could graph the dimensions, calculate areas and notice the patterns, and hence explore elementary calculus ideas like limits, maximum/minimum and descriptive functions. This will encourage relational understanding and, as the conceptual network develops, students will have opportunities to reflect on and articulate their own contributions to the learning process.

There are some excellent open questions that are readily available, such as those in Sullivan & Lilburn (1997). However, it is not difficult to adapt most textbook questions by giving the students a problem that includes what is traditionally the answer, and getting them to write the possible questions. For example:

Traditional question	*Open question*
50c + 20c + $2	I have three coins in my pocket that total $2.70. What might the coins be? How many different combinations are possible?
300 + 450 + 625	A farmer had 1375 sheep and three paddocks. What are some possible combinations?
Round off (n)	A number has been rounded off to 5.6. What might the number be?

Conclusion

Children are born with the capacity to love mathematics, to understand it as a network of changing, meaningful and useful ideas, and to become aware of themselves as knowers of mathematics. A lot of the experiences they have in our classrooms do not have these objectives in mind.

I have argued for the use of strategies that allow students to develop four levels of understanding. The first level involves understanding objects and the traditional ways that these are manipulated. The second level, growing out of experience with objects, is understanding of symbols and the ways that these are generally manipulated. The third is the realm of abstracted, connected and increasingly complex concepts. The fourth is a domain of understanding of self as an empowered knower and rational user of mathematics. These levels are not sequential—they develop in parallel—and perhaps represent levels of reflective abstraction more than of conceptual development.

The result of maths teachers providing experiences that encourage such abstraction should be many more of Queensland's students spontaneously saying: "I do understand. I know it. It is beautiful. I love it."

References

Mousley, J. A. (1990a). Organised chaos: Problem-posing in a constructivist classroom. In K. Milton & H. McCann (Eds.) *Mathematical turning points: Strategies for the 90s* (pp. 60–72). Hobart: Australian Association of Mathematics Teachers.

Mousley, J. A. (July, 1990b). *Organising for learning in a problem-based learning classroom*. Paper presented to the annual conference of the Mathematical Education Research Group of Australasia, Hobart.

Mousley, J.A. (1990c). Towards Fractals: P-12. In M.A. Clements (Ed.), *Whither mathematics?* (pp. 39–43). Melbourne: MAV.

Schoenfeld, A. (1986). On having and using geometric knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge*: The case of mathematics (pp. 225–263). Hillsdale, NJ: Erlbaum.

Skemp, R.R. (1988). Goals of learning and qualities of understanding. *Mathematics Teaching 79*, (1), 44–49.

Sullivan, P., & Lilburn, P. (1997). *Open ended maths activities. Using "good" questions to enhance learning*. Melbourne: OUP.

Sullivan, P., & Mousley, J. (July, 1999). *Thinking teaching: Seeing an active role for the mathematics teacher.* Paper presented to the annual conference of the Taiwan Association for Research in Mathematics Education, Taipei.

Mathematics and God

On the night of November 23, 1654, from 10:30 to about 12:30, Pascal experienced a religious ecstasy which caused him to abandon science and mathematics for theology. The result was the writing of the Lettres provinciales and the Pensées; for only one brief period, in 1658–1659, did Pascal return to Mathematics. One night in 1658 toothache or illness prevented him from falling asleep, and as a distraction from the pain, he turned to the study of the cycloid. Miraculously, the pain eased, and Pascal took this as a sign from God that the study of mathematics was not displeasing to Him.

from *The History of Mathematics, Carl B. Boyer.*

Investigating Exponential Growth and Decay with Graphing Calculators

Janelle Ford and Michael Cheng
The University of Queensland

This is the second of three articles resulting from seminars on technology-based mathematics teaching given by Dip. Ed. students at The University of Queensland under the guidance of Merrilyn Goos. The first article and some background information were published Volume 24 No. 1 of Teaching Mathematics, pages 16-18. The third article will be published in Volume 24 No. 3.

Introduction

Exponential growth and decay appears in the Mathematics B Syllabus within the topic Exponential and Logarithmic Functions and Applications (Board of Senior Secondary School Studies, 1992). This activity is based upon activities found at the Texas Instruments web page (Texas Instruments, 1998a), which offers a collection of classroom activities for the TI-xx series of calculators. Our activity has been created from ideas and materials developed by Jerry Staniszewski, Cindy Panke and Charles Hofman (Texas Instruments, 1998b). Modifications were made to these original activities in order to make a coherent whole; in addition, a few ambiguous questions were removed, and a question concerning the unsuitability of the Y=0 data point was added.

Although this activity was written with the TI-83 graphing calculator in mind, it may be adapted to other calculators provided they have the capacity to produce scatter plots and perform the exponential regression analysis of the data.

Materials

- *M & Ms* are the item of choice (for the obvious reason that the students can eat their data when they finish), but see the section on adaptations (below) for alternative items.
- Containers (e.g. paper cups, mugs)

M & Ms aren't always practical or possible in a classroom. Suggestions for alternative items: e.g. buttons, playing cards, coins. For a greater challenge, use objects which produce a non-binary outcome: e.g. dice, or the alignment direction of scattered paddle pop sticks. Here, the procedure would be modified, for example, by removing all the dice showing a particular number.

Depending on the item chosen and the availability of calculators, this activity may be done in small groups or individually. It may even be demonstrated by the teacher, with all students then using this one set of data.

Discussion Points

- The meanings of *a* and *b* in the equation
- The use of derived exponential equations to make predictions
- Significance of the r^2 and r values for the fit
- Radioactive decay, carbon dating, growth of bacteria and other life-related situations which follow this simple model.

Adaptations

To increase the challenge of this activity, use it as an introduction to equations involving indices. That is, the activity could be used to demonstrate that the equations they have learned/used so far (linear, quadratic, etc) are inadequate to model these phenomena. After students have used the calculators to graph the scatter plot, but before they start trying to fit all the regression functions possible, a discussion of possible models would be useful. Students could argue for their particular model and reason why it fits the data and is appropriate to the model. The various models could then be fitted to the data (using the graphing calculators) and a comparison made.

Investigating Exponential Growth and Decay with Graphing Calculators

Materials: *M & Ms*TI-83; Graphing calculator; small paper cup.

Experiment 1: Decay

1. Begin this experiment with a cupful of *M & Ms*. Count the number of *M & Ms* and write this as trial 0.
2. Shake the cup and pour out the *M & Ms*. Remove those with the *M* showing.
3. Record the number remaining in the table below. Repeat this procedure until only one *M & M* remains. **NB** If the number of *M & Ms* reaches zero at any trial, the experiment is over. <u>Do not</u> use the zero result as part of your data.

Trial number	No. of *M & Ms*
0	
1	
2	
3	
4	
5	
6	

Using your TI-83

4. Enter the number of trials into List 1.
5. Enter the number of *M & Ms* that were face up into List 2.
6. Produce a scatter plot via the STAT PLOT menu.
7. Using the STAT menu and CALC sub-menu, find the exponential function, $y = a\,(b)^x$, which best fits the data. Write the equation.

   ………………………………………………...
8. Graph this equation over the scatter points.

Questions

1. Explain the coefficients a and b in the equation.

   ………………………………………………...

   ………………………………………….……
2. If I started with 900 *M & Ms*, which decayed at the same rate, predict how many trials I would have to do to reach one *M & M* ……………………..

Experiment 2: Growth

1. Begin this experiment with four *M & Ms* in a cup. Shake the cup and pour out the *M & Ms*.
2. Note the number of *M & Ms* having the *M* showing. Add this number of new *M & Ms* to the cup.
3. Record the new total of *M & Ms* in the table below. Repeat this procedure five more times, using the new total each time.

Trial number	No. of *M & Ms*
0	
1	
2	
3	
4	
5	
6	

Using your TI-83

4. Clear the lists from the previous experiment.
5. Enter the number of trials into List 1.
6. Enter the number of *M & Ms* that were face up into List 2.
7. Produce a scatter plot via the STAT PLOT menu.
8. Using the STAT menu and CALC sub-menu, find the exponential function, $y = a(b)^x$, which best fits the data. Write the equation:

   ………………………………………………...
9. Graph this equation over the scatter points.

Questions

1. Predict the number of *M & Ms* on trial 9: …………………………….
2. Predict the number of trials needed to have 300 *M & Ms*: …………………
3. Explain the coefficients a and b in the equation.

   ………………………………………………...

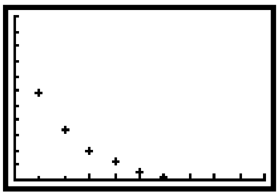
   ………………………………………………...

Sample Solution

Experiment 1: Decay

Trial number	No. of *M & Ms*
0	50
1	29
2	17
3	10
4	6
5	3
6	1

The scatter plot of this data and exponential regression analysis are displayed below (Figures 1 and 2), together with the graph of the regression equation superimposed over the scatter points (Figure 3).



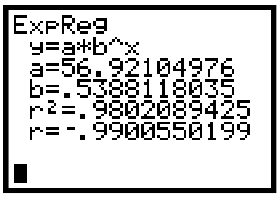


Figure 1. Scatterplot of decay experiment data

Figure 2. Exponential regression for decay data

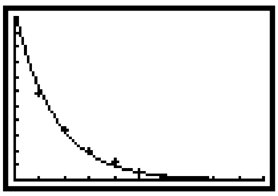


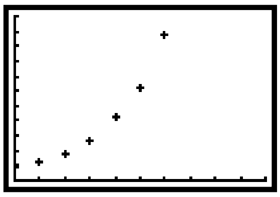
Figure 3. Regression line for decay experiment data

The equation for the decay has the form $y = a(b)^x$, where a represents the initial number of M & Ms and b the rate of decrease (equal to the probability of the M showing, or ½). For the sample data above the regression equation was $y = 56.92\ (0.54)^x$. Hence, if I started with 900 M & Ms which decayed at the same rate, the number of trials to reach one M & M can be found by solving the equation $y = 900\ (0.54)^x$ for $y = 1$. Alternatively, a graphical solution can be produced with the calculator by plotting the function and exploring it via the TRACE button, or by setting up a table of values via TBLSET and TABLE.

Experiment 2: Growth

Trial number	No of *M & Ms*
0	4
1	6
2	9
3	13
4	21
5	31
6	49

The scatter plot of this data and exponential regression analysis are displayed below (Figures 4 and 5), together with the graph of the regression equation superimposed over the scatter points (Figure 6).



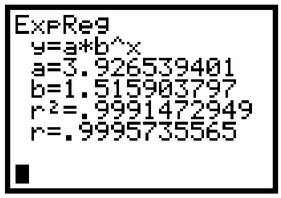


Figure 4. Scatterplot of growth experiment data

Figure 5. Exponential regression for growth data

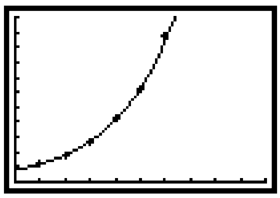


Figure 6. Regression line for growth experiment data

The equation for the growth has the form $y = a(b)^x$, where a again represents the initial number of M & Ms and b the rate of increase (equal to 1.5 because the initial amount increases by the probability of M showing). For the sample data above the regression equation was $y = 3.93\ (1.52)^x$.

References

Board of Senior Secondary School Studies (1992). *Senior Syllabus in Mathematics B*. Brisbane.

Texas Instruments (1998a). *http://www.ti.com*

Texas Instruments (1998b). http://www.ti.com/calc/docs/activities.htm

Page 15 Problem

Solution to 'Game Show'

The contestant is better off changing.

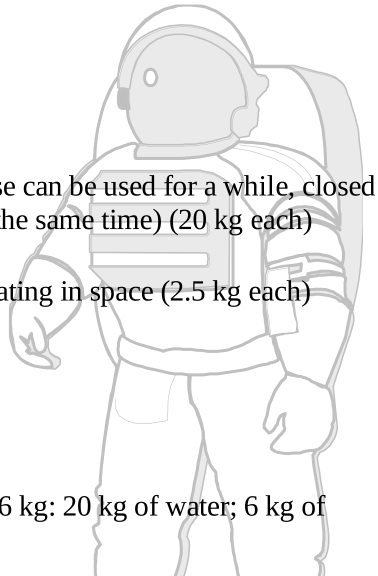
Suppose the contestant can play a large number of games and suppose she never changes. She will pick the correct curtain in one third of the games. She will thus win the car one third of the time.

Suppose now that she changes every time. In one third of the games she will choose the correct curtain first up and will thus change to an incorrect curtain and lose. In the other two thirds of the games, she will pick one of the incorrect curtains first up. The compare will then order the other incorrect curtain to be opened, leaving her the correct curtain. So on each of these two thirds of the games, she will win. She will thus win two thirds of the time.

Moonwalk

Marcus and Anastasia have landed on the moon 100 km from base. They cannot contact base and base does not know where they are. So they have to walk there. The ground is fairly rough but walkable. They can average 3 km/hr but the walking is tiring - they need at least 8 hours rest after a 10 hour walk. The sun is just setting. The nearly full earth is fairly high in the sky. Their space suits have enough power to keep them at the right temperature for 80 hours. They each need to drink 2 litres of water and eat 1kg of food every 24 hours. They also need to take some sort of map of the terrain to help them find the way to the base.

They have the following items they can take.



- First Aid Kit (4 kg)
- Inflatable Life Raft (18 kg)
- Hot Air Balloon (70 kg)
- 12 tanks of oxygen each enough to last one person 20 hours (these can be used for a while, closed up, then finished later, but they cannot be used by two people at the same time) (20 kg each)
- Fireworks (28 kg)
- Eight 1.5 kg packs of onion-flavoured sponge cake suitable for eating in space (2.5 kg each)
- Detailed map of the moon on computer disk (200 g)
- Battery-powered computer (6 kg)
- Battery for computer (3 kg)
- 6 cartons of Coke (9 kg each)
- Firewood (15 kg)
- Two 20-litre containers of water suitable for drinking in space (26 kg: 20 kg of water; 6 kg of container; the container can be partly emptied before you start)
- Box of waterproof matches (100 g)
- Siren (2 kg)
- Piece of metal that fell off the space capsule (48 kg)

Neither of them can carry more than 60kg. They can make it there alive. How do they do it?

Level of mathematical knowledge required: Year 5.
Some basic astronomical knowledge is helpful, though not essential.

Algebra as the Study of Relations

David Ilsley
Queensland School Curriculum Council

If you want to cheer yourself up, go out onto the street and ask the first person you see about their experience of school algebra. Then watch their eyes light up as they explain to you exactly what algebra is, what it is for, how much they enjoyed learning it and how much difference knowing it has made to their life.

To many students, algebra seems to be a collection of skills without any obvious connection between them except that they all involve letters. As teachers, we can explain that algebra is 'generalised arithmetic', but such an explanation is too abstract to be of help to students.

This article suggests an approach to the teaching of algebra in which algebra is seen as the art of using relations. Students can easily grasp the idea of a relation and the usefulness of relations is more obvious than is the usefulness of manipulating letters.

All the concepts and skills involved in algebra are introduced as means of getting information from relations. Thus students can see the purpose of every new idea and skill as it is introduced.

Background

Traditionally, algebra has been taught by introducing the idea of a pronumeral, like x, then operations on pronumerals to produce expressions like $3x + 2$, before leading on to collecting like terms, expanding, factorising, performing inverse operations etc. To most students, this algebra bore no relation to anything else they met in life. They took it on faith that the knowledge was useful – and for most students, the faith was misplaced because they would never use it except to pass an algebra test.

It is not surprising that when you ask many adults what algebra is about, they cannot give a meaningful answer.

More recently, a different approach has become popular – one in which students observe patterns, then use algebraic expressions to describe those patterns. This approach makes algebra slightly more meaningful for beginners, but the problem is that it begins with some fairly difficult concepts (like writing a formula to describe the pattern 4, 7, 10, 13, 16, . . .), and doesn't lead on naturally to other parts of algebra – after this different introduction, algebra courses then settle back into the old routine of learning about pronumerals, expressions etc. Also, although the sequences of match-stick patterns typically used are more 'concrete' than the pronumerals and expressions of the old approach, they still don't relate very clearly to anything else that might happen in the student's life.

In this article I am going to suggest a different approach. This approach begins with and is based on everyday phenomena; it develops relatively easy concepts before going on to more difficult ones; it develops the concepts which are most likely to be of use in everyday life first; it can be developed progressively from the early grades of primary school; and it leads very naturally into the algebra of senior and tertiary mathematics.

The approach is based entirely on the concept of a relation, and, in this sense, the study of algebra can be regarded as the study of relations.

Introducing relations

Many people would have trouble explaining what algebra is, but the concept of a relation is one that people can grasp fairly easily.

Consider the following information which shows how much Family Payment is received per fortnight by families on $30 000 per year combined income.

A family with no children receives $0
A family with 1 child receives $22.70
A family with 2 children receives $56.85
A family with 3 children receives $161.95
A family with 4 children receives $274.55
A family with 5 children receives $387.15
A family with 6 children receives $499.75

This is the relation between the number of children in a family and the Family Payment received. It is a relation.

Here is another relation – the relation between time and outdoor temperature:

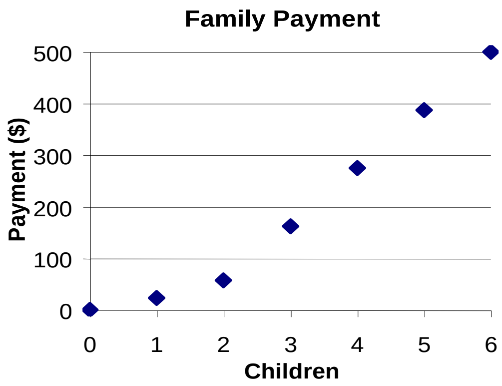
Time	Temperature
5 a.m.	17°
6 a.m.	17°
7 a.m.	18°
8 a.m.	20°
9 a.m.	23°
10 a.m.	25°
11 a.m.	26°
12 noon	27°

A relation is information which allows us to find the value of one quantity given the value of another.

The first example allows us to find the Family Payment, given the number of children, or to find the number of children, given the Family Payment; the second example allows us to find the temperature, given the time, or the time(s), given the temperature.

Relations can be presented in a number of ways:

- as a verbal statement (as in the Family Payment example)
- as a table (as in the temperature example)
- as a graph, eg. the following graph for the Family Payment example



- as a set of ordered pairs: eg. (5, 17), (6, 17), (7, 18), (8, 20), (9, 23), (10, 25), (11, 26), (12, 27) for the temperature example.

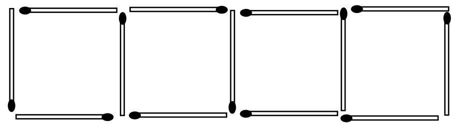
Primary school students have no trouble understanding, dealing with, using and producing such information. They should be able to master the skills of obtaining one quantity given the other in any form of a relation and of converting from one form of a relation to another, eg. from a table to a graph or vice versa. Practice at this and awareness that they are dealing with relations should provide a good grounding for the more in-depth study of relations in secondary school.

Examples of relations that might be looked at in primary school are:

- Number of children vs no of teeth lost
- Number facts known vs week
- Mass vs height for the children in a class
- Price vs number of sausage rolls
- Hunger vs time
- Boredom vs time
- Height vs age for a plant
- Temperature vs time
- Maximum temperature vs day
- Time taken vs distance run

Relations with patterns

Consider two relations: one between maximum temperature and day of the month for the first 6 days of a month; and one between the number of adjacent match squares and the number of matches required to make them for 1 to 6 squares.



These relations are given in table form below.

Day	Max temp
1	27
2	24
3	25
4	27
5	28
6	22

No of squares	No of matches
1	4
2	7
3	10
4	13
5	16
6	19

Looking at these relations, students can readily tell that they can predict the number of matches for 7 squares, but that they cannot predict the temperature on the 7th day. This leads to the idea that some relations have a pattern and some do not. Relations with patterns can be expressed in a very concise way – by a formula. In the case of the match-stick example the formula is

number of matches = number of squares × 3 + 1

Abbreviations for variable names can be introduced as a way of making formulae even more concise:

$$\boldsymbol{m = s \times 3 + 1}$$

Seeing letters in algebra as abbreviations removes much of the mystery which often surrounds the use of pronumerals. The use of abbreviations, however, might be left until the students themselves see a need for them. There is no problem with initially using formulae with the variable names written out in full.

At this stage students should be given formulae, but should not be expected to generate them themselves.

Discrete and continuous relations

When students present relations as graphs, the issue of discreteness vs. continuity soon arises – students will argue about whether the points should be joined up. This issue can be dealt with at this stage. Students should realise, for example, that a continuous relation can be presented as a graph and perhaps a formula, but not fully as a table or set of ordered pairs. A sample of ordered pairs needs to be used.

Equations

From their early experiences with relations, students should have no trouble finding one variable given the other, if the relations are expressed as verbal statements, tables, graphs or sets of ordered pairs. New skills are needed, however, when the relation is in the form of a formula.

If the formula is explicit with regard to the dependent variable, then finding the value of the dependent variable from the value of the independent variable is the process of substitution. Substitution would be, for most people, the most common use of formulae, and it is appropriate that this skill should be mastered before the other skills of formal algebra.

Substituting for the independent variable in a formula with two variables produces an *explicit* equation in one unknown:

$$m = s \times 3 + 1$$
$$\Downarrow$$
$$m = 4 \times 3 + 1$$

'Solving' this equation is straightforward.

On the other hand, substituting for the dependent variable in a formula with two variables produces an *implicit* equation in one unknown:

$$m = s \times 3 + 1$$
$$\Downarrow$$
$$13 = s \times 3 + 1$$

Such implicit equations are what we normally call 'equations' and much of algebra consists of learning techniques to solve these.

There are four main ways to solve equations.

- Guess and check
- Backtracking
- Doing the same thing to both sides (or the balance method)
- Graphing

These are given in order of increasing conceptual difficulty.

The first three can be taught at this stage. Students master 'guess and check' quite easily, though they can get frustrated with the laboriousness of it. They are then happy to learn the backtracking method. As the name implies, the backtracking method involves working backwards to the value of the unknown. For instance, to solve $(3x - 5)^2 = 49$, we draw the following:

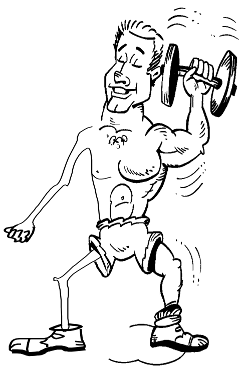
$$\underset{x}{\boxed{}} \xrightarrow{\times 3} \boxed{} \xrightarrow{-5} \boxed{} \xrightarrow{\square^2} \boxed{49}$$

then work backwards filling in the squares as we go.

$$\underset{x}{\boxed{4}} \xrightarrow{\times 3} \boxed{12} \xrightarrow{-5} \boxed{7} \xrightarrow{\square^2} \boxed{49}$$

$$\div 3 \qquad\qquad +5 \qquad\qquad \sqrt{}$$

Students can solve quite complex equations easily with this method, provided that they are familiar with inverse operations and the conventions regarding order of operations and provided the unknown occurs only once in the equation. They should use this method for quite a while before going on to 'doing the same thing to both sides', which is really just a modification of backtracking, but which can handle a wider range of equation types.



This fellow exercised the left half of his body.

It shows what can happen if you don't learn to do the same thing to both sides.

When the equations become more complex, equation solving provides a meaningful context for manipulation techniques, which, traditionally, tend to be taught in isolation. These manipulation techniques include collecting terms, expanding, operating on rationals etc.

At this point students can be introduced to the skill of solving problems by writing and solving equations, both via formulae and directly. This skill is a great asset in senior mathematics and science as well as in several contexts in junior maths. Trigonometry is an example. Solving a problem like '*If something costs $14.50 after a 30% reduction, how much did it cost before?*' is another example.

Formulae from tables and graphs

By now, students should have had experience in converting between tables and graphs and in converting from formulae to tables and graphs, but not in converting tables and graphs to formulae. Learning to convert linear relations to formulae can be achieved by getting to understand the relation between gradient and y-intercept and the parameters m and c in the general form of the linear relation $y = mx + c$. Along with this, students can learn to rearrange formulae into standard form and to change the subject of formulae.

Functions

The concepts of a function as a special and particularly useful type of relation and as a sequence of operations are generally introduced in junior algebra. This can be extended to introduce the most common families of functions, *viz.* linear, reciprocal, power, exponential, quadratic and perhaps polynomial and sinusoidal functions. For each family, students should become aware of some situations in which it arises, its standard form (the idea of a parameter can be useful here), the general shape of the graphs and methods for solving equations derived from the functions.

Facility with solving equations by graphing with a graphics calculator allows students to solve equations derived from functions of any family, and indeed more complex functions. A case might even be made for using graphing as the primary means for solving quadratic equations.

Explaining why some things always happen

A final application of algebra, which is not generally given a lot of attention in junior mathematics, but which is emphasised in *The National Statement on Mathematics* is its use to explain things like the following:
Pick a date on a standard calendar.

S	M	T	W	Th	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

Add together the date above it and the date below it. Then add together the date to the left of it and the date to the right of it. Are the results the same? Will this always happen? Why?

The explanation lies in the relations between the sums and the original date.

Let the original date be d.
Then the first sum a is given by

$$a = d - 7 + d + 7,$$

which simplifies to $a = 2d$.
The second sum b is given by

$$b = d - 1 + d + 1,$$

which simplifies to $b = 2d$.

Leading in to senior and tertiary algebra

Algebra and calculus in Maths B and C is based on the idea of a function. A study of relations like that described above, would provide a good foundation for this.

The algebra required in Maths A is restricted to

- substituting into formulae;
- writing and solving equations, eg. to solve trigonometry problems;
- graphing linear functions in linear programming.

These skills are dealt with early on with this approach.

In tertiary mathematics, the concept of a relation is fundamental. It is viewed primarily from the ordered-pair perspective, but the thorough study of relations from all the perspectives above will provide a solid foundation for these further studies.

A teaching sequence

The table on the next page shows one possible learning sequence which incorporates the above ideas.

Students could do A1 and possibly even A2 in primary school. The decision when to do A2 will depend on when students need to be able to use formulae. [My preference is to leave A2 and the use of formulae till high school.]

In high school, more able students could cover A2 to A9 over years 8 to 10. They would then be well prepared for Maths B and C in Year 11.

Average ability students could cover A2 to A5 over years 8 to 10. They would then be well prepared for Maths A in year 11.

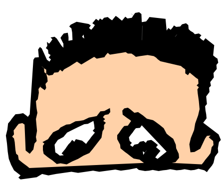
Lower ability students could cover A2 over years 8 to 10. This would enable them to use basic algebra that is useful in everyday life.

POSSIBLE LEARNING SEQUENCE FOR ALGEBRA

A1	a) Know what a relation is b) Understand how a relation can be expressed as a verbal statement, as a set of ordered pairs, as a table and as a graph c) Find the value of one quantity given the value of the other in a relation d) Understand the terms 'variable', 'independent' and 'dependent' e) Convert a relation from one form to another f) Tell whether a relation has a pattern and extend such relations by adding ordered pairs
A2	a) Understand that relations with patterns can be expressed as formulae b) Know what is meant by an equation c) Substitute in a formula and solve the resulting explicit equation d) Convert from a formula to other forms of a relation e) Substitute into formulae with more than one independent variable f) Treat discrete and continuous relations appropriately g) Handle algebraic shorthand
A3	a) Solve implicit equations by guess and check b) Solve implicit equations by backtracking c) Solve implicit equations by doing the same thing to both sides d) Collect terms and expand brackets
A4	a) Write and solve equations (via formulae and direct)
A5	a) Understand gradient and axis intercepts b) Convert between tables, graphs and formulae using $y = mx + c$ c) Graph linear relations using axis intercepts d) Rearrange formulae
A6	a) Understand what is meant by the domain and range of a relation b) Understand a function as a special type of relation and as a sequence of operations c) Use the language of functions
A7	a) Factorise linear expressions b) Collect factors c) Explain why some things always work
A8	a) Solve single equations by graphing b) For each of the following families of functions, know some applications, the general form, the shape and methods of solving equations: proportional, linear, reciprocal, power, quadratic, polynomial, exponential, sinusoidal, step and absolute value
A9	a) Solve simultaneous equations by substitution b) Solve simultaneous equations by graphing c) Operate on and cancel rational expressions

Psychotic: someone who thinks 2 + 2 = 5

Neurotic: someone who knows that 2 + 2 = 4 but worries about it



REVIEWS

Instant Graphics Calculators Lessons - Precalculus

Instant Graphics Calculators Lessons - Calculus

Sue Thomson
Ian Forster

Addison Wesley Longman
1998

$65 Each

For those already familiar with the books of instant lessons produced by these authors here is another pair of high quality reference books to add to your bookshelf.

With the availability and acceptance of calculator technology becoming more wide spread, this pair of books is ideal for those who need access to quality materials that will work in their classroom. The activities are produced as blackline masters, ready for immediate use in the classroom or in the training of teachers.

The activities are written with an aim, a time requirement to complete the activity, a list of prerequisite knowledge and any special calculator techniques required to complete the tasks. These are extremely useful and are a result of much intense and thorough trialing of the materials. The answers to the activities are also provided in the teachers notes.

The teacher notes are clear and concise and provide well considered advice for those teachers not completely at ease with the technology. The activities have been carefully chosen so as to cater for a broad range of student ages and abilities.

The tasks provide opportunities for students to work both independently and in groups and are mathematically rich learning experiences. Tasks included provide many challenges for students and provide some good opportunities for developing a deeper understanding of the mathematics involved.

The books include advice for the use of the current models in both the Casio (CFX 9850G) and Texas Instrument (TI-83) ranges. This involves a "Getting started with…" for each of these models. While these pieces of advice are included, the activities could just as easily be used with any of the other graphics calculator models currently on the market without any cutting and pasting being required.

The Pre-Calculus book includes activities relating to straight lines, quadratics, polynomials, circles, exponential curves, trigonometric functions, absolute value functions, intersecting lines and curves, moving graphs and inequalities.

The Calculus book includes activities relating to the gradients of functions, the nature of stationary points, graphs of functions and their gradients, optimisation problems, trigonometric and exponential functions, integrals and log functions.

I would have no hesitation in recommending these books as a reference text for any maths department looking at incorporating Graphics calculator technologies into their programs.

Gary O'Brien
Cannon Hill Anglican College

Queensland Mathematics Workbook 2

Hellyn Goodman
Neville Goodman

Magraw - Hill

This is the second in a series of workbooks designed for Queensland Schools while being lined up against the National Statement for Mathematics. It has been written with year 9 students in mind, to be used in conjunction with a textbook.

The book contains activities designed to maintain and develop basic skills, along with problem solving tasks and open-ended investigations. The authors suggest that the book be used during class, as homework and as part of the formal assessment process. Based on my initial exploration of this book I would concur with this suggestion.

The activities are grouped into the 5 strands: number, measurement, space, chance and data and algebra. The activities are graded from level 4 through to level 7, with each exercise concentrating on a particular skill which has been identified as an indicator of a particular level within that strand.

REVIEWS

As a result of the structure of the book it would not be appropriate to just use it in the order it has printed, it would require the teacher to decide on the relevant activities to complete. The relevant activities may in fact vary from student to student. An easy-to-use topics links page is included to assist in this process.

The book is set up such that students would purchase their own copy of the book and record all work in the book. The book is set out with space for working and encourages students to explain what they are doing and what they have discovered.

The activities are well constructed and as a result, students' understanding is developed well. The book contains a student profile where successful completion of activities can be recorded against the levels of the National Profile for Mathematics. For schools that report using these levels, this book may provide much valuable information.

The twelve open-ended investigations are well constructed, interesting and varied in nature. They provide many opportunities for students to adopt different approaches and to develop good problem solving skills.

The marking schemes supplied would be very useful in grading these activities as they link possible student responses to statement levels. The answers to the activities are not included in the text (do they think the students may copy them??) but are provided in an accompanying booklet.

I was impressed by the layout, presentation and activities included in the book, but to adopt this as a workbook for an entire year level would require the commitment of the school to adjust their work programs to effectively incorporate this and to include the newly available information in their assessment programs.

Lydia Commins
Ipswich Girls Grammar School

MathsMania Year 7

Andrew Boswell

McGraw-Hill

This recent publication is one of a series of student workbooks written to support the Queensland Syllabus. According to the Preface, a teacher's resource book of black-line masters is also available to support the program.

The book is organised into 36 units and it is suggested that each represent a week's work. The activities on each page are well spaced and easy to read. The use of two colours and amusing drawings enhances the visual appeal.

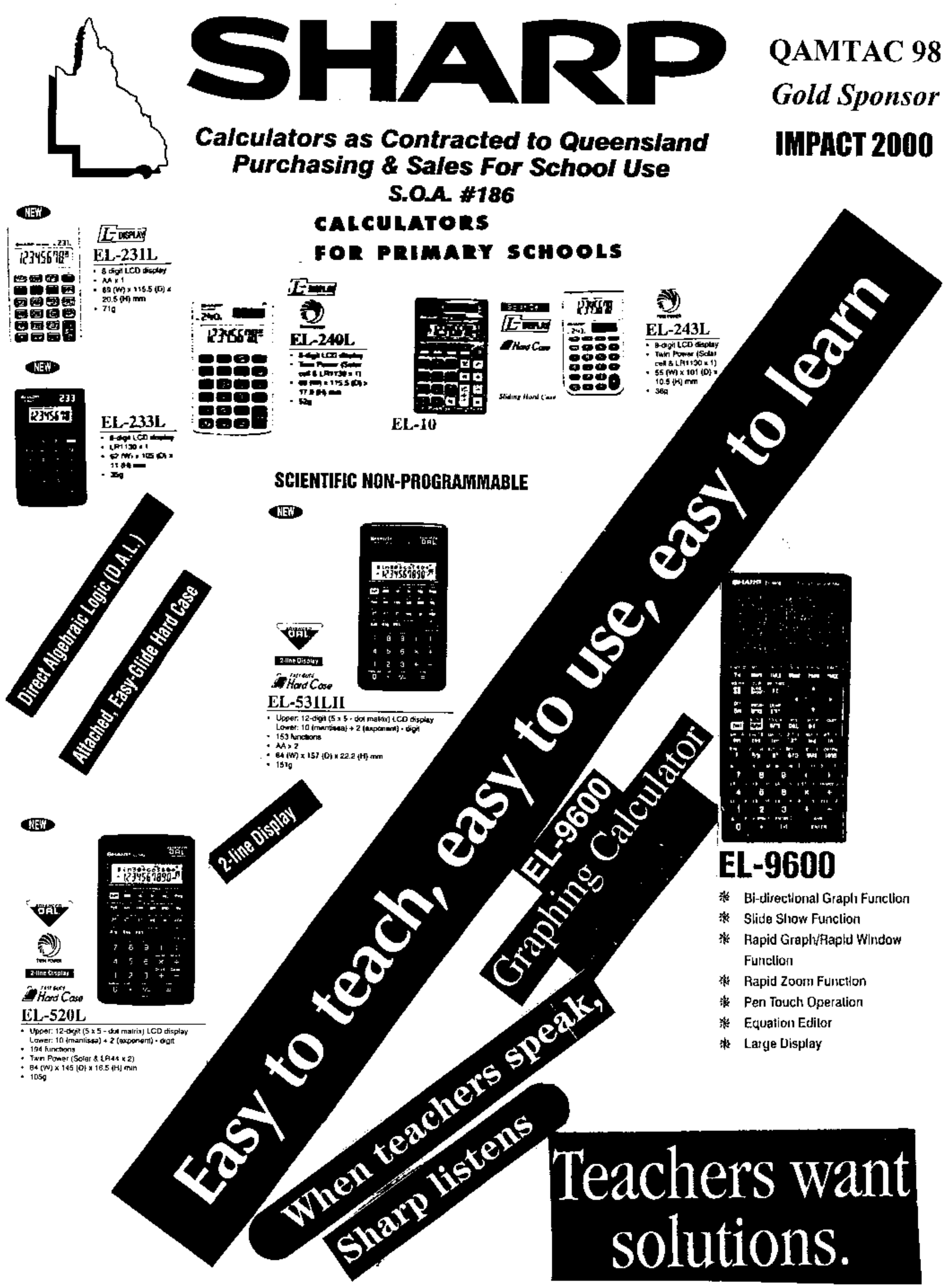
The activities are cross-referenced to the focus statements in the Year 7 Sourcebook. This indicates that it could be used to supplement a teaching program based on the sourcebook.. Although the preface asserts that the books contain a balance of traditional style activities and problem solving and investigative approaches, far more space is devoted to traditional activities.

Within each unit there is an emphasis on written number work, particularly the development of the written algorithm, to the extent that the activities suggested for the strands other than number are mostly trivial and lack context or suggestion of an investigative approach. The numeration activities are clearly set out and well sequenced, but the operations concentrate on the standard written forms with little space devoted to choosing the operation or working in context. No consideration is given to encouraging collaborative learning, discussion or mental work. Strategies used for estimating answers are not alluded to.

Although vocabulary lists are given, the activities generally do not demonstrate the use of the given words or provide any context to demonstrate their syntactical use. Similarly, the use of concrete materials is suggested but not clearly demonstrated or emphasised.

This book could be used to supplement a teaching program which uses the techniques and approaches outlined in the sourcebook, but, if used alone, could lead to a very narrow approach to the teaching and learning of mathematics.

Jill Hedley
Jamboree Heights State School



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Slide Show Function
Rapid Graph/Rapid Window Function
Rapid Zoom Function
Pen Touch Operation
Equation Editor
Large Display
When teachers speak, Sharp listens
Teachers want solutions.

Spreadsheets

and upper primary mathematics

Annette R. Baturo & Tom. J. Cooper
Centre for Mathematics and science Education
QUT

* This paper is based on a project funded through a DEETYA 1997 Higher Education Innovative Programs grant.

Most teachers today adopt the pedagogical philosophy that students need to construct knowledge, that is, they need to be active participants (both physically and cognitively) in the teaching/learning process. Constructing knowledge promotes understanding through the development of rich internal (mental) representations of the knowledge and its connections to other knowledge (Hiebert & Carpenter, 1992). Having rich interconnected schema, which Baturo (1998a) refers to as *structural knowledge*, enables students to see isomorphic relationships between tasks such as adding whole numbers, decimal fraction, common fractions and measurement, and between mathematical domains (e.g. fractions and probability). This ability to see relationships therefore facilitates transfer of knowledge as well as promoting remembering because of the reduced cognitive load.

One fruitful area of research into how students come to understand some aspect of mathematics has focused on the construction of different knowledge types and their connections (e.g. Baturo, 1998a; Leinhardt, 1988; Payne & Rathmell, 1977; Sfard, 1991). For example, Baturo's model incorporates four knowledge types, namely, *entry knowledge*, *representational knowledge*, *procedural knowledge*, and *structural knowledge*. Structural knowledge (i.e. connected schema) should be the endpoint of formal instruction. This paper provides some spreadsheet activities that are designed to promote the various types of knowledge.

Two powerful teaching strategies, *patterning* and *reverse teaching*, are used in these activities whenever practicable. Looking for patterns requires students to observe at least three examples, abstract those features that are the same and those that are different, predict what the next term in the pattern will be, and then validate their prediction. For many numerical patterns, a calculator or spreadsheet can be used for validation.

Reverse teaching means ensuring that the bi-directional nature of relationships is explored. For example, to construct enriched fraction schema, whole→part and part→whole activities (see Figure 1) should be provided. Another example of reverse teaching is having students calculate the area of a given rectangle as well as having students construct a rectangle with a given area.

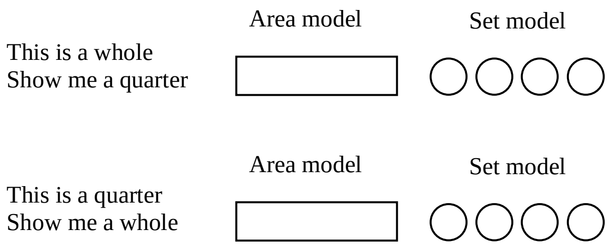
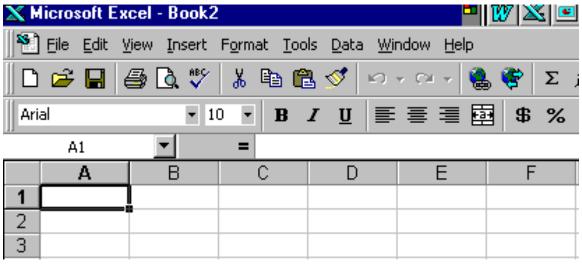


Figure 1. Example of a reverse teaching activity.

Electronic spreadsheets have three particular features that make them useful teaching tools: (a) an array of rows and columns in which static data (numbers or shapes) can be inserted; (b) a "number crunching" ability; and (c) the ability to present numerical data in a variety of graphic forms. Thus electronic spreadsheets can be used to develop representational, procedural, and structural knowledge.

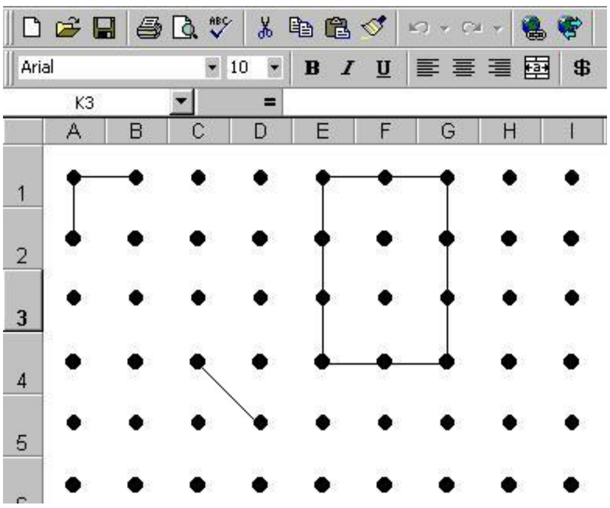
Task 1: Constructing an electronic geoboard (Baturo, 1998b)

Setting up. Open *Excel*. The screen shows a worksheet comprised of rows and columns (i.e. a grid). The columns are labelled A, B, C, ... (left to right); the rows are labelled 1, 2, 3, ... (top to bottom). A particular cell is denoted by its column letter and row number, so that D3 refers to the cell in the fourth column from the left and the third row from the top. If you place your cursor in a cell and click, you can type data in the cell.



Set up the rows and columns to form squares. To change the column widths, place the cursor on the grey cell marked A, hold down the Shift key and move the cursor across to H (or as far as you like). Activate the Format menu, highlight Column then Width (on the right), and enter "5". To change the row heights, place the cursor on "1", hold the shift key, and move the cursor down to Row 10. Activate the Format menu, highlight Row, then Height, and enter "30".

Construct the electronic "nails". In Cell A1, draw a small circle (see the following picture) and position it in the middle of the cell. Highlight the circle, copy it (from the Edit menu or from the Copy icon on the formatting toolbar), place the cursor on A2, hold the shift key as you drag the cursor to A10, and press the Paste command (from the Edit menu or the Paste icon on the formatting toolbar). You should see circles ("nails") in each cell in the first column. Highlight the column of circles, press copy, place your cursor in B1 and press Paste, then place your cursor in C1 and press Paste. Continue until you have a large enough grid.



Remove the gridlines so children can focus on the intervals between the nails by activating the Tools menu and highlighting Options. Click on Gridlines to deactivate the lines. The electronic geoboard is now ready to draw lines and shapes of various dimensions.

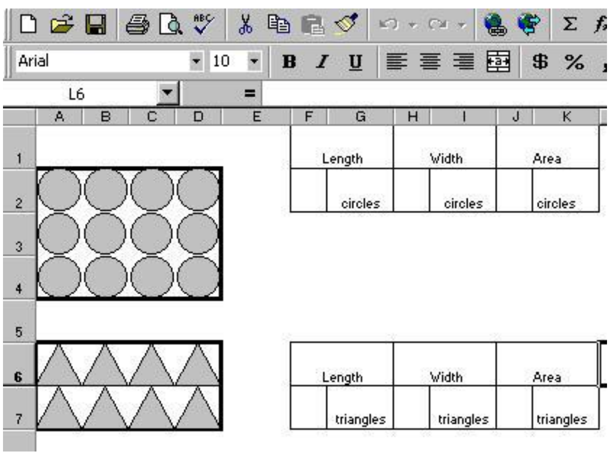
Activities.

- Have students free draw to start with to become familiar with the drawing tool.
- Ask them to draw some open shapes and some closed shapes. Ask: *Which shapes have an area? What could you do to the other shapes so they have an area*?
- Ask the students to draw rectangles to particular dimensions (e.g. 6 units long, 3 units wide). Ask students to record (pencil-and-paper) the area of the shapes (e.g. 18 sq. units, 12 sq. units, etc).
- Ask students to draw a variety of polygons and their diagonals (encourage irregular as well as regular polygons) and then do the reverse, namely, start with diagonals and construct the resulting polygons.

Task 2: Developing the formula for the area of a rectangle (Baturo, 1998b)

This task uses both the array and number processing features of the spreadsheet. Its purpose is to facilitate students' "discovery" of the formula for calculating the area of rectangles. Two activities are provided here: (a) using regular nonstandard units of measure (e.g. circles, triangles), and (b) using regular standard units of measure (i.e. squares)

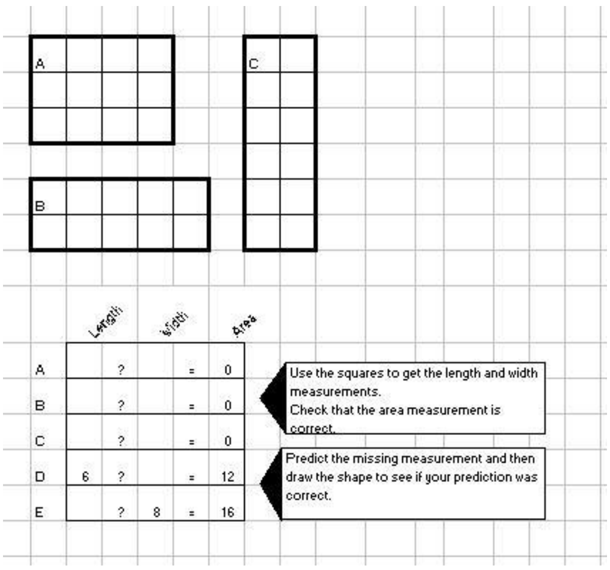
Construct electronic grids for measuring a rectangle. The worksheet shown in the following picture can be developed from the same processes as those used to construct the electronic geoboard – just use different shapes instead of the small circles that were used to represent the "nails " on the geoboard.



Construct the formula for calculating the area. To enter the formula in J2 for calculating area, first place the cursor on J2. Type = to activate the formula, then place the cursor on F2, type *, place the cursor on H2, press Ente*r*. [Alternatively, type =F2*H2 and press *Enter*.] When students enter numbers in F2 and H2, the formula is applied and thus the area is calculated.

Pattern searching. Discuss with students how the area number may have been calculated. Have them verify the area measurement by counting the units of measure. Discuss which of the units of measure (circles or triangles) is better for measuring the area of a shape.

The second activity is set up in exactly the same way. However, this activity goes a step further and has students validating their predictions.

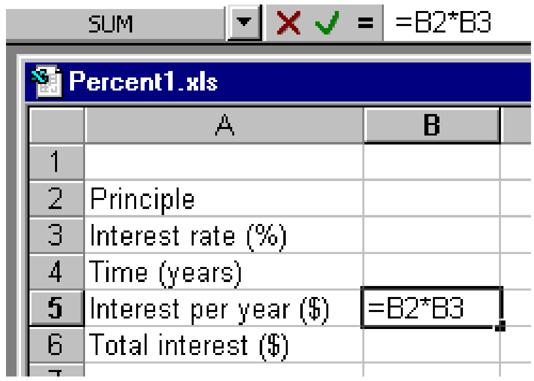


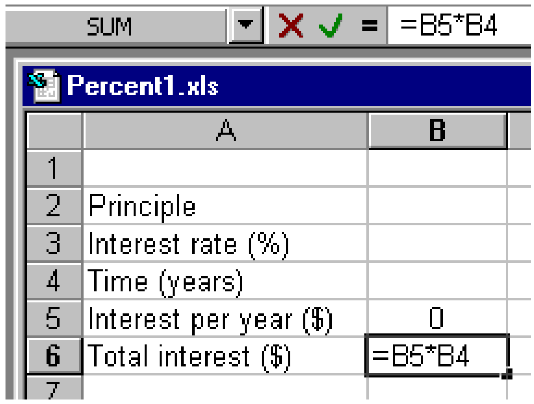
Task 3: Exploring hire purchase (Baturo, 1998c)

Prerequisite knowledge. Students should have a sound understanding of percents, and interest.

Problem. What interest is paid on a loan of $2 500 for 3 years at an interest rate of 7%?

Setting up the Excel program. List the “knowns” first and then the “unknowns” (see the following pictures). Leave the unknowns in Column B blank for the students to complete.





Place the cursor on B (Column B), click and Copy. Place your cursor on C and drag the cursor across to Column H, go to *Format/Edit* and select *Paste Special/All*. The formulae you entered in Column B are now copied to the other columns. This means that when the students change any one of the “knowns”, all calculations are automatically adjusted. Taking away the drudgery of the calculations frees the student to focus on the effects of the change.

Using the program. Before having students enter data on the computer, ask them to develop a pencil-and-paper table in which they list all the knowns then all the unknowns, then have them enter the knowns data into the *Excel* program. (Let them work in groups.)

Discussion. Is the interest amount more or less than expected? How might you check whether the computer has calculated correctly?

Investigations.

- Discuss what would happen if the principal was changed to a smaller amount. Would the total amount of interest to be repaid be more or less? Why? Have students enter smaller amounts in Columns C, D, and E and note what happens. Repeat for larger principals.
- Similarly, predict then verify what would happen if: (a) the interest rate was changed; and (b) the

amount of time was changed.

- Investigate how many years it would take, at a particular rate, for twice the principle to be repaid.
- Investigate which is the better deal when only one or two variables (i.e. principle, rate, time) are changed.

Note: Students could simply overwrite the data in Column B but then the previous record is lost. However, overwriting the data is much more dramatic because the change is instantaneous. To offset the problem of losing the previous records to refer to for comparison, one of the students in each group could act as scribe (i.e. constructing a pencil-and-paper table, entering the original data and the changed data.

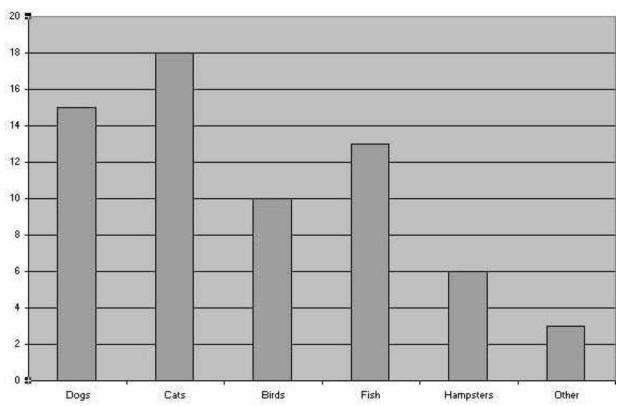
Task 4: Graphing with spreadsheets (Baturo, 1998b)

Enter the collected data on a spreadsheet (see the following picture).

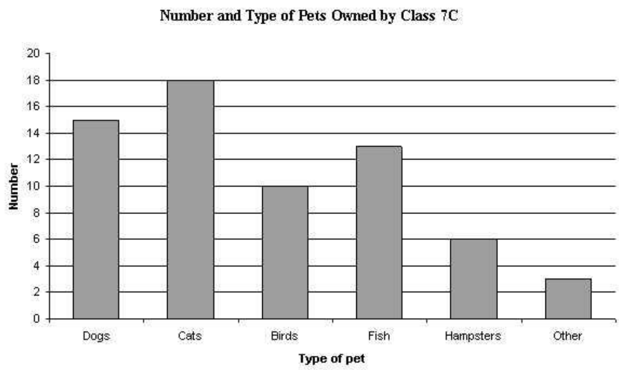
Pet graph.xls

	A	B	C
1	Dogs	15	
2	Cats	18	
3	Birds	10	
4	Fish	13	
5	Hampsters	6	
6	Other	3	

Activate the graphing utility. The easiest way to change the data to a graph is to highlight the two columns (pet type and number of each type) and then press the F11 key. This will convert it to a vertical bar graph (see the following picture).

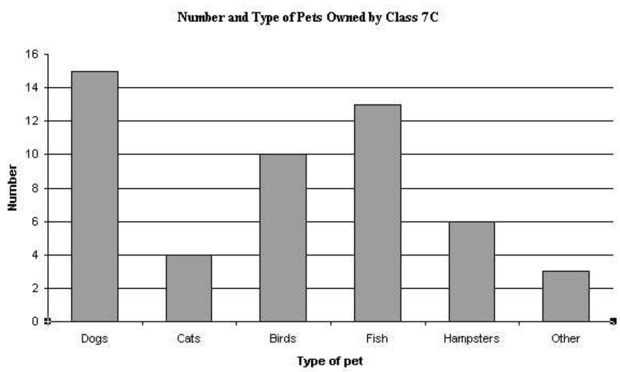


Titles will need to be added and the graph's characteristics (e.g. font type and size, colours) can also be altered. To add titles and remove or change the legend, highlight the *Chart* menu and then *Chart Options* in the menu. This provides a range of options that you can take for the graph's presentation. In the following picture, the graph and axis titles have been altered and the legend has been removed. The background shading has been altered as well. This was done by placing the cursor on the graph's background and going to the *Format* menu and selecting *Selected Plot Area*. The font used in the graph's title was changed by selecting the title and then going to the fonts showing on the toolbar. The font for the vertical axis was changed by placing the cursor on the axis and clicking. This activates a menu that enables a range of formatting to be done. Follow the same procedure for the horizontal axis.



The graph form can be changed to a horizontal bar graph, a circle graph, or a divided bar graph (and many others). To change the graph's form, go to the *Chart menu* and select *Graph Type*. Select from the range of graphs presented. Make changes to the graph's characteristics as described above.

Re-graphing with spreadsheets. To change the data in the original graph, the easiest way is to return to the original data (Sheet 1 in *Excel*). For example, changes were made to the number of cats (from 18 to 4). These changes were immediately transferred to the graph (see the following picture).



Extension. Have students experiment with the graphing utility; and observe the relations between the data and the resulting graphs. Ask students to construct a graph with no titles to give to their classmates to construct the data, embed the data in a problem, and write the missing titles.

Discussion

Mathematics is rich in patterns and having students look for patterns is a worthy teaching strategy. However, to be effective, students need to be encouraged to articulate the pattern, predict what the next term in the pattern will be, and then have some means of validating it. Calculators and spreadsheets are particularly useful in enabling students to validate many numerical patterns.

Mathematics is also rich in relationships and, to develop sound structural knowledge, both directions of the relationships need to be explored. Thus reverse teaching should be part of the teacher's repertoire of teaching strategies.

Electronic spreadsheets facilitate pattern predicting and validating, and an understanding of the bi-directional nature of relationships.

References

Baturo, A. R. (1998a). *Year 6 student's cognitive structures and mechanisms for processing tenths and hundredths*. Unpublished doctoral dissertation, Centre for Mathematics and Science Education, Queensland University of Technology, Brisbane, Australia.

Baturo, A. R. (1998b). Developing concepts and skills. **in** T. J. Cooper & A. R. Baturo (Eds.), *Using technology to develop numeracy* (pp. 1-43, Resource Module 2). Brisbane, Qld: QUT printery.

Baturo, A. R. (1998c). Applying percent knowledge. **in** T. J. Cooper & A. R. Baturo (Eds.), *Using technology to develop numeracy* (pp. 12-24, Resource Module 3). Brisbane, Qld: QUT printery.

Hiebert, J. & Carpenter, T. (1992). Learning and teaching with understanding. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning (pp. 65-97)*. New York: Macmillan.

Leinhardt, G. (1988). Getting to know: Tracing students' mathematical knowledge from intuition to competence. *Educational Psychologist*, *23*(2), 119-144. Hillsdale, NJ: Lawrence Erlbaum.

Payne, J. & Rathmell, E. (1977). Numeration. **in** J. Payne (Ed.), *Mathematics learning in early childhood*. (1977) Yearbook of the National Council of Teachers of Mathematics. Reston, VA: NCTM.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.

The Queensland Senior Syllabuses Roots of Conflict?

Stephen Norton QUT sj.norton@qut.edu.au
Denis Bridger St. Rita's College dmgbridger@bigpond.com.au

Introduction

The intention of this article is to stimulate discussion on the nature of the Queensland Senior Syllabuses in Mathematics A, B and C (Board of Senior Secondary School Studies, 1992). These documents are currently under review and now is an appropriate time to discuss some of the underpinning and somewhat unstated principles which drove the 1992 versions. The article examines the philosophies that have influenced our syllabuses. There are indications of a mismatch between secondary teachers' beliefs and values with respect to teaching mathematics and teaching strategies promoted by syllabus documents. In California, the tension from this mismatch has found expression in the "Math War" (Stevenson, 1999). There are indications that situations similar to those found in Californian exist in Queensland and have the potential to give rise to conflict that could hinder the growth of diverse teaching strategies in secondary school mathematics.

The Queensland Syllabuses – History and Roots of Conflict

The Queensland Senior Syllabuses (BSSS, 1992) have adopted a rationale which reflects the National Council of Teachers of Mathematics Standards (NCTM, 1989) which emphasise the importance of learning to reason mathematically, problem solving, mathematical communication, exploration of mathematical ideas within a cultural context and inquiry-based learning. There is contention among mathematics educators regarding the appropriateness of the educational reform movement towards inquiry-based learning.

Inquiry-based learning is characterised by active involvement in doing mathematics and investigation to facilitate personal development within a rich exploratory environment (Ernest, 1991). The inquiry-based teacher is a facilitator and stimulator of learning, managing the classroom so students are challenged to think and reflect on the validity of their existing concepts. Students are engaged in exploring and formulating ideas that are often self-generated. That is, mathematics is integrated into constructive activities (Cobb, Yackel, Wood, & McNeal, 1992) and seen as an evolving social practice (Cobb, 1988; von Glaserfeld, 1987). The claimed strength of this approach to teaching and learning is that it embraces the social and cultural nature of mathematics learning.

The inquiry-based approach to learning mathematics has been sponsored by Australian Educational bodies. For example, the National Statement on Mathematics for Australian Schools (Australian Education Council, 1990), although careful not to direct teachers to adhere to particular approaches, emphasised developing students' capacity to use mathematics in "dealing with non-routine mathematical problems and unfamiliar situations … both individually and collaboratively" (p. 12). Statements such as "too great an emphasis on predictable mathematical tasks may undermine the capacity of students to deal with unfamiliar tasks" and "mathematics develops through the interaction of communities of people" (p. 13), promote a constructivist approach to learning mathematics. Criticism of traditional approaches to mathematics teaching is thinly veiled:

> The systematic and formal way in which mathematics is often presented conveys an image of mathematics which is at odds with the way it actually develops … School mathematics should show the intuitive and creative nature of the process (p. 14).

The document goes on to recommend that "Learners construct their own meanings from, and for, the ideas, objects and events which they experience" (p. 16).

The Queensland Senior Syllabuses in Mathematics A, B, and C (BSSS, 1992) has embraced the AEC (1990) constructivist recommendations. For example, the Syllabus in Mathematics B (BSSS, 1992, p. 1) advocated an approach to mathematics study through "problem solving and life related contexts", and did not include "more traditional learning experiences" (p. 11) in recommended learning experiences. Although, it did not mention the term "inquiry-based" learning, it strongly encouraged teachers to use an "investigative approach." For example, ten of the seventeen recommended learning experiences on page 13 of the syllabus ask teachers to get their students to investigate or request student constructions. Unfortunately no where in the document is a definition of what is meant by the term "investigative" explained. However, to many educationalist these statements indicate a desire on the part of the syllabus authors for teachers to move away from traditional instruction and towards approaches encouraging the construction of mathematical knowledge within a cultural context by using inquiry-based learner-focused teaching strategies.

The approach to teaching mathematics in the Senior Syllabus in Mathematics B (BSSS, 1992) has been supported by the journal of the Queensland Association of Mathematics Teachers Inc. (QAMT), *Teaching Mathematics*. An examination of recent QAMT issues indicates that most of the contributors seem to support a move towards inquiry-based learning. For example, in Volume 23 No 3, Stillman (1998) advocated group work on modelling activities, Ilsley (1998) presented activities in which students could use spreadsheets to simulate and experiment with mathematical models, Nason (1998) recommended that students engage in mathematical investigations, and Boyce (1998) suggested that in problem solving several different solutions could emerge from group interactions. All these contributors have in essence advocated an inquiry-based or investigative approach to teaching in the activities that they described.

Rejection of Inquiry-based Learning

The inquiry-based approach to learning has not been universally accepted by mathematics teachers either in Australia or the United States. In Australia, the attempts by curriculum writers to persuade teachers to adopt teaching strategies consistent with the underpinning construcitvist learning theories have met with resistance (McDonald & Ingvarson, 1995; McRobbie & Tobin, 1995; Norton, 1999). In the United States, the opponents of inquiry-based learning in mathematics teaching have formed a movement called *Mathematically Correct* (see Web site http://mathematicallycorrect.com). The intensity of debate between the movement and supporters of inquiry-based learning has been so strong that it has been termed the "Math War." Supporters of the mathematically correct approach have termed curricula based upon constructivist learning theories and inquiry-based learning "new new math" and "fuzzy math" (Prawat, 1997). They have been able to get a new policy, titled *The Mathematics Framework for the California Public Schools K-12*, (California Department of Education, 1998) approved in draft form in December 1998 by the California State Board of Education. Various observers have argued that endorsement of this framework could lead to increased emphasis on direct instruction and repetitive practice and greater opposition to problem solving (e.g., Becker & Jacob, 1998).

Opponents of the constructivist and inquiry-based approach to the teaching and learning of mathematics reject the idea that student construction of knowledge ought to be cental to mathematics learning (Allen, 1998; Loveless, 1997). They have reported that student construction downgrades basic skills, partly because the time taken to "discover" means less material can be covered and partly because "memorisation" has not been developed and utilised (Allen, 1998; Clopton, 1998; Loveless, 1997; Quirk, 1998). They have reported that mathematics is a discovery that models the reality of the world, but opposed student discovery where students "construct their own more accommodating versions of reality" (Loveless, 1997, p. 2). These beliefs are reflected in their recommendations for a return to teacher exposition or "direct instruction" with an emphasis on basic skills and practice with students working predominantly individually (Allen, 1998; Klahr & Thomas, 1998; Quirk, 1998; Tashman, 1994; Watkins, 1999). They have reported that mathematical concepts and understandings are better and more efficiently developed in students through teacher-centred pedagogy rather than constructivism-based learner-centred teaching strategies, putting them at odds with supporters of inquiry-based learning regarding how students learn and consequently which teaching strategies are most appropriate.

Conclusion

It is possible that Australian states like Queensland, which have underpinned their senior mathematics syllabuses with constructivist learning theories, will experience the same "grassroots rebellion" (Loveless, 1997) or "backlash" (Becker & Jacob, 1998) as in California. Many Queensland teachers have not embraced the philosophy and strategies articulated by the current syllabus, nor do they want to (Norton, 1999). Their differences with the syllabus authors, and many of the *Teaching Mathematics* contributors, are deeply rooted in their own knowledge and beliefs. A number of authors have observed that knowledge and beliefs have powerful restraining influences upon teachers by acting as a filter through which teachers view and interpret their own and others' models of teaching (e.g., Alexander, 1996; Thompson 1992).

It is apparent to the authors of this article that the philosophy and intention of the 1992 versions of Mathematics A, B, and C have not been universally well received by secondary mathematics teachers. That there has not been more criticism of its attempts to direct teachers towards a particular style of teaching may have a number of explanations. Here is one possibility, the process of moderation of assessment submissions conducted by peers has failed in its attempts to ensure that the philosophies of the syllabuses have been implemented. Part of the reason for this situation may be that the syllabuses have not been well understood by many teachers and not been liked by others. The point is: Why do we continue to have a syllabus that has a rationale and recommended learning strategies that are either poorly understood or not well respected?

The current senior syllabuses are under review. The authors of this article encourage you to express your thoughts to the BSSSS and make constructive recommendations.

References

Alexander, P. A. (1996). The past, present, and future of knowledge research: A reexamination of the role of the role of knowledge in learning and instruction. *Educational Psychologist, 31*(2), 89-92.

Allen, F. B. (1998). Repairing school mathematics in the US. Mathematically Correct [http://mathematicallycorrect.com/ report.htm] (7/1/99).

Australian Education Council (1990). *A national statement on mathematics for Australian Schools*. Carlton, Curriculum Corporation.

Becker, J. P., & Jacob, B. (1998). Math war developments in the United States (California) [Http://www.math.ethz.ch/EMIS/mirror/IUM/ICMI/bulletin/44/MathWar.html]. The International Commission on Mathematical Instruction (ICMI) (6/01/99).

Board of Senior Secondary School Studies (1992). *Senior Syllabus in Mathematics B*. Brisbane. The Author.

Boyce, R. (1998). More ideas for developing skills in CIII. *Teaching Mathematics, 23*(3), 32-35.

California Department of Education (1998). *Mathematics Framework for California Public Schools K-12*. [http://www.cde. ca.gov/cilbranch/eltdiv/mathtw.htm] (7/01/99).

Clopton, P. (1998). Testimony to the United States House or Representatives Committee on Education and the Workforce [Http://206.86.183.194/mc/hredcomm.htm]. (Mathematicallycorrect). (7/01/99).

Cobb, P. (1988). The tension between theories of learning and instruction in mathematics education. *Educational Psychologist*, *23*(2), 87-101.

Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for research in Mathematics Education, 23*(1), 2-33.

Ernest, P. (1991). *The philosophy of mathematics education.* London.: The Falmer Press.

Ilsley, D. (1998). Solving differential equations with spreadsheets. *Teaching Mathematics 23*(3), 11-14.

Klahr, D., & Thomas, T., Sokol. (1998). Carnegie Mellon researchers say direct instruction, rather than "discovery learning" is best way to teach process skills in science [Http://www.eurekalert.org/ releases/direct-sciskill.html] (7/1/99).

Loveless, T. (1997). The second great math rebellion [Http://www.teachermag.org/ew/vol-1//0/love.hI/]. Teacher Magazine on the Web (6/1/1999).

McDonald, H., & Ingvarson, L. (1995). Free at last? Teachers, computers and independent learning. Annual meeting of the American Educational Research Association. San Francisco. California: (EDRS Document Reproduction ED 389 278).

McRobbie, C., & Tobin, K. (1995). Restraints to reform: The congruence of teacher and student actions in a chemistry classroom. *Journal of Research in Science Teaching, 32*(4), 373-385.

Nason, R. (1998). Mathematics investigations: knowledge building in the mathematics classroom. *Teaching Mathematics 23*(3) 20-23.

National Council of Teachers of Mathematics (1989). *NCTM Standards: Introduction.*

[http;//groper.enc.enc.org/reform/journals/ENC 2280/nf_2801.htm]. (2/12/98).

Norton, S. (1999). *Secondary mathematics teachers' responses to the potential of computers in their teaching*. Unpublished doctoral dissertation, Queensland University of Technology.

Prawat, R. S. (1997). Fuzzy math, old math, and dewey [Http://www.teachermag.org/ ew/vol-//16 prawat.ht/]. Teacher Magazine on the Web (6/01/99).

Quirk, W. G. (1998). *The anti-content mindset, the root cause of the "Math Wars"* [Http://www.wgquirk.com/content.html] (21 10 98).

Stevenson, H. W. (1999). *Professor Harold W. Stevenson on the NCTM Standards* [Http://mathematicallycorrect.com/hwsnctm.htm]. Mathematically Correct. (7/01/99).

Stillman, G. (1998). Including social critique in mathematical learning. *Teaching Mathematics 23*(3), 3-7.

Tashman, B. (1994). *Our failure to follow through* [Http://darkwing.uoregon.edu/~adiep/ft/tashman.htm] (7/1/1999).

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 127-146). New York: National Council of Teachers of Mathematics.

Von Glaserfeld, E. (1987). Learning as a constructive activity. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 3-18). Hillsdale, NJ: Lawrence Earlbaum Associates.

Watkins, C. L. (1999). Follow through: Why didn't we? [Http://darkwing.uoregon.edu/ ~adoep/ft/watkins.htm] (7/1/99).

Apathetic: someone who doesn't care what 2 + 2 equals

Megalomaniac: someone who thinks 2 + 2 = 879

Financial Education For Our Children

by David Rosenberg

When I completed senior, my knowledge of finances generally was very poor.

I studied an academic course, including maths and science, but there were no practical studies, even in simple subjects, such as how to open a bank account or do a budget - nothing.

Our parents help us out, as best they can, and it is a moot point about whose responsibility it is to teach children financial management. The problem is, most parents need help themselves and do not necessarily have the skills to teach their children financial management.

So off we went into the big bad world, earning money and doing the best we could. Life was simple when I first started work, earning $32 a week. I paid $10 board, $12 for food and a few extras, and saved $10 to buy a stereo and put a deposit on my first car.

Then life got more complicated. I paid a deposit on a car and the balance was on hire purchase. By that time, I was earning $75 per week but living in a unit and going "out on the town". I often ran out of money before the next pay day. Still, bills were paid and, if I needed to eat, I could go to my parents.

Times have certainly changed and our financial lives are far more complicated. When I first started work in the early 70s superannuation was not what it is today. You could have any type of life insurance you wanted as long as it was whole of life or endowment. An investment was a term deposit and shares were misunderstood, and used generally by the wealthy.

"Most of us, I suppose, learnt as we went along..."

Most of us, I suppose, learnt as we went along and, in times of high inflation, the mere fact of home ownership increased our net worth.

Times have changed and therefore our children's financial education should not be left to chance.

How can we help?

Financial Technology has more than 100 clients in the teaching profession. They include university professors, lecturers, secondary and primary teachers, both private and public. Our advisers tell us that many of these educators suggest financial planning should be taught in schools.

We agree. We now participate in a schools program called "Mastering Your Money". The program is an eight-section lecture program aimed at teaching the basics of personal financial planning. Even though the program is aimed at year 11 and 12 students, it has been well received by TAFE colleges and even those in retirement.

The reason the program is so popular is that so many of us have forgotten the basics.

The program can be tailored to a one-hour lecture, or a full, two-day course, depending on the situation.

The program is not limited to schools and can be used by companies as a benefit to employees.

If you think we can be of assistance at the school your children attend, the school at which you teach, or the company at which you work, please give us or your adviser a call so we can provide further details.

PYTHAGORAS REVISITED
Lessons from the Past

Paul Dooley, Fairholme College, Toowoomba. e-mail pmdooley@mathgym.com.au

This article is the second in a series of four in this year's Teaching Mathematics. The author is attempting to show how historical/cultural/social/philosophical contexts can be used in the teaching of some pretty familiar, and sometimes dry mathematical concepts to secondary students. The author has found that adolescents show a keen interest in this approach , as it gives them some understanding of their place in space in time. The articles which appear in this Journal are shortened extracts from a collection of Essays and Activities on the authors web site http://www.mathgym.com.au To read a more expansive treatment and to obtain a copy of any of the Activity sheets - Click on the "history" button at the site and then follow the link to the Activity you want. You can then print out the page to get a black-line master for reproduction.

Introduction

The Pythagorean view of the universe rested squarely on the belief that Natural (counting) number was the key to the various qualities of mankind and matter. Since, in their view, everything was composed of number, the explanation for an object's existence could only be found in number. Elsewhere about this time, number existed for utilitarian purposes only, as a device for solving problems in calendar construction, building and commerce.

It was the Pythagoreans who saw number as important in itself, the numbers themselves having "personality in a rustic landscape". They are credited with making the distinction between **logistic** (art of computation) and **arithmetic** (number theory). Kline [1] quotes the famous Pythagorean Philolaus (425 B.C.), as writing:

"Were it not for number and its nature, nothing that exists would be clear to anybody either in itself or in its relation to other things...You can observe the power of number exercising itself ... in all acts and the thoughts of men, in all handicrafts and music."

Pythagoras and the early Brotherhood initially treated number concretely, as patterns with pebbles, but over time the Pythagoreans developed and refined their concept of number into the same abstract entity which still exists today. Though it is difficult to separate fact from fancy in some of the surviving references to the Pythagoreans, it is generally conceded that they began number theory, and were responsible for the introduction and development of number mysticism in Western Society.

A. Early concepts of Number and Number Mysticism

Number mysticism is not generally associated with "serious mathematics" but from the early Pythagoreans until the 19th century many venerated mathematicians practised some forms of numerology. In more recent times this was in return for patronage from the influential and aristocratic circles who sought some mystical assistance in their daily endeavours. Even today, existing Western cults refer to Pythagorean doctrines on number. According to Boyer [2]:

"Many early civilisations shared various aspects of numerology, but the Pythagoreans carried number worship to its extreme, basing their philosophy and their way of life upon it."

To the Pythagoreans, each number possessed its own special attributes. See for example the table below.

Number	Property of the number
1	**monad** (unity) generator of numbers, the number of reason
2	**dyad** (diversity, opinion) first true female number
3	**triad** (harmony = unity + diversity) first true male number
4	(justice, retribution) squaring of accounts
5	(marriage) = first female + first male
6	(creation) = first female + first male + 1 ?
10	(Universe) tetractys

Fortunately, in developing their number mysticism, the Pythagoreans also valued rigour and proof. To this end they searched for the essential properties and definitions of many numbers. The following is a brief description of their ideas about number.

Definitions of Unity

Some of the early definitions of unity are found in Heath [3] where the **monad** is described as *"limiting quantity"*, or as *"the common part or beginning of how many"*.

Definition for Number

Similarly, there were many different definitions for number, the Pythagorean being essentially: "Number is a collection of units".

Definition for Odd/Even Numbers

The Pythagoreans made the distinction between **odd** and **even** numbers as seen in this early definition (Iamblichus in Heath):

*"An **even** number is that which admits of being divided, by one and the same operation, into the greatest and the least (parts), greatest in size but least in quantity* (i.e. two lots each half size) ... *while an **odd** number is that which cannot be so treated, but is divided into two unequal parts"*

Many superstitions became associated with the odd and even numbers. For example the odds were considered masculine and divine while the evens were considered feminine and thus earthly and human.

Definitions for Prime/Composite Numbers

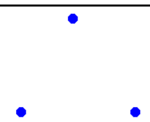
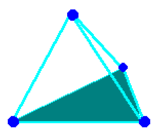
Early Pythagoreans described what we would call **prime** numbers as *"prime and incomposite numbers"* and it was generally accepted that a **prime** number was *"measured by no number, but by an unit only"*, though there was some disagreement about whether 2 was prime. The **composite** numbers were those which were *"measured by any less number"*. In keeping with the Pythagorean view of number as a pattern of stones, prime numbers were also called **rectilinear** because they can be represented as a single line of stones only, compared to the composites which can be also be arranged into equal numbered groups of stones. **Composites** were further distinguished as **plane** or **solid**. Heath quotes Theon of Smyrna (1st century A.D) as writing:

"of composite numbers they call those which are contained by two numbers plane, as being investigated in two dimensions and, as it were, contained by a length and a breadth, while (they call) those (which are contained) by three (numbers) solid, as having the third dimension added to them."

There is an activity on this work on the web-site.

The tetractys

To the Pythagoreans the holiest number of all was the number 10 or the **tetractys**. In addition to their other "personalities" the first four numbers had a special significance in that their sum accounted for all the possible dimensions:

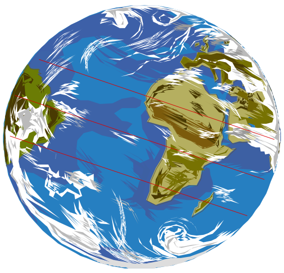
Number	Geometric property	Geometric shape
1	generator of dimension 0	
2	line of dimension 1	
3	triangle of dimension 2	
4	tetrahedron of dimension 3	

Summing the first four numbers they got 1+2+3+4 = 10. Since these were the only numbers that were needed to demonstrate all known objects (geometrically) then the sum of all these objects, that is the sum of these numbers, was believed to represent the known Universe. The properties of the **tetractys** still has persuasive influence in mystic cults of today. Some argue that it was the Pythagorean veneration of the **tetractys**, not so much the number of digits on hands or feet, which is responsible for our present use of the base ten.

In addition to the **tetractys**, the Pythagoreans developed other concepts of "fourness" in nature such as the material elements of earth, air, fire, and water.

A Model for the Universe

Since the **tetractys** was the number for the Universe, the Pythagoreans believed that there had to be 10 heavenly bodies. In addition to the visible earth, sun, moon and five planets, they added a central fire and a "counter-earth" on the opposite side of the central fire. This initial attempt at explaining cosmology in terms of mathematical principles is the foundation of our present models for the Universe.



Definitions for Amicable Numbers

The Pythagoreans, on Iamblichus' (in Heath) somewhat doubtful authority, are credited with discovering **amicable** or **friendly** numbers. Two numbers are **amicable** if each is the sum of the proper divisors (that is all the divisors except the number itself) of the other. For example 220 and 284 are **amicable** since the sum of the proper divisors of 220 are 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284 and the sum of the proper divisors of 284 are 1 + 2 + 4 + 71 + 142 = 220. Superstition maintained that two talismans bearing this pair of numbers would seal a perfect friendship between the wearers. The pair also became significant in magic, sorcery, and astrology.

Definitions for Perfect/Deficient/Abundant Numbers

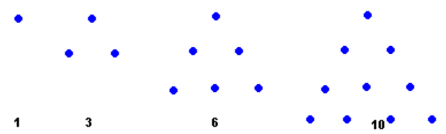
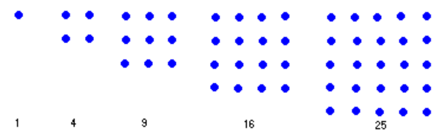
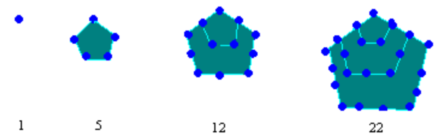
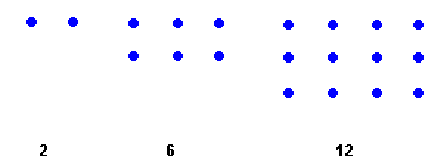
Other numbers which were given mystical powers and associated with the Pythagoreans were the **perfect**, **deficient**, and **abundant** numbers. A number is **perfect** if it is equal to the sum of its proper divisors, is **deficient** if its sum falls short of the number, and is **abundant** if the sum exceeds the number. 6 is a **perfect** number (1 + 2 + 3), 8 is **deficient** (1 + 2 + 4), and 12 is **abundant** (1 + 2 + 3 + 4 + 6).

B Figurate Numbers

To Pythagoras and the early Pythagoreans, number was "atomistic", it existed as bundles of a fundamental elementary object - unity. Numbers were represented as patterns formed by collections of these units. These patterns were formed into the so-called **figurate** or **polygonal** numbers, the triangular, square, pentagonal, oblong, etc. In fact when we call numbers "figures" today we are still using the jargon of the Pythagorean Brotherhood.

We have already looked at some of the properties of the square numbers in the earlier article.

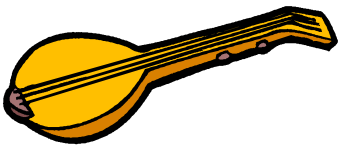
Most figurate numbers begin with the number generator 1. From there the next number is determined by the number of pebbles needed to make the desired pattern as in the table:

Figurate Number	**Geometric shape**
triangular numbers	
square numbers	
pentag-onal numbers	
oblong numbers	

The Oblong numbers are those that can be arranged in a rectangle one unit wider than it is high - each is twice a Triangular number. The Oblongs have sides in the ratios 1:2, 2:3, 3:4, 4:5, 5:6,....

C Number patterns in Music

Readers who are familiar with the theory of music will recognise these as the intervals in decreasing order of consonance: Octave, Perfect Fifth, Perfect Fourth, Major Third (4:5), Minor Third (5:6), etc. It is Pythagoras who is credited with discovering this mathematical relationship between music and number.



This discovery, that the pitch of a note is related to the length of the string which produced it, is credited as being the spark which ignited Pythagoras' imagination and philosophy. It allowed Pythagoras a glimpse of a whole new order in the Universe, one governed by intellect and logic and capable of the sublimest of pleasures. And it seems that a glimpse was all that he needed. With this discovery, Pythagoras and the Pythagoreans set in train a way of investigation which has proved to be one of the most productive ideas in human history – that mathematics can be used to unravel the mysteries of the Universe. It was, in the words of Koestler [4] :

"..epoc-making: it was the first successful reduction of quality to quantity, the first step towards the mathematization of human experience - and therefore the begining of science..."

There is an activity on this work on the web-site.

The Music of the Spheres

The Pythagoreans wove their musical discoveries into their mathematical cosmology to produce a hauntingly beautiful description of the Universe. The Pythagorean Universe consisted of a central, spherical earth surrounded by the heavenly objects. These were attached to crystal spheres at distances determined by the regular solids (solids which can be circumscribed by a sphere). The rotation of these spheres produced wondrous musical harmonies.

The Pythagoreans explained that normal people cannot hear the "harmony of the spheres" because they have grown too accustomed to hearing it from birth (Pythagoras alone was supposed to be able to hear it). Nonetheless the quest for the mathematics behind these harmonies captivated some of the greatest minds over the next two thousand years. Koestler describes how the early physicist Kepler spent much of his productive life trying to discover the harmony of the spheres. The three laws he is best remembered for were virtually footnotes to his investigations into the harmonies.

D Incommensurables

The early Pythagorean belief that "all was (Natural) number" meant that they did not consider fractions as numbers. What we consider as fractions today were seen by the early Pythagoreans as ratios of Natural numbers as in the musical scales above, not as numbers in their own right. It was fundamental to the beliefs of the early Pythagoreans that all things (abstract, natural, human affairs) were understandable by considering the intrinsic properties of Natural numbers and their ratios. This was the foundation of the faith and scholarship of Pythagoras and essential to this faith was the concept of **commensurability**. Heath quotes the first scholium on Book X of the Elements as stating:

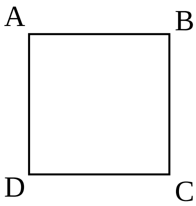
"..the Pythagoreans were the first to address themselves to the investigation of commensurability, having discovered it by means of their observation of numbers... They called all magnitudes measurable by the same measure ***commensurable****"*

For example, the number 8 can be ***measured*** in twos - the oblong number 4×2, 12 can be also ***measured*** in twos - the oblong number 6×2, so 8 and 12 are commensurable as they both can be ***measured*** in twos. Similarly with geometry, they reasoned that given any two line segments of unequal length, it should always be possible to find a third line segment, perhaps very small, that can be marked off a whole number of times into each of the given segments. With the concept of commensurability, the early Pythagoreans could confidently base their faith on "all is number". If any pair of numbers were found to be incommensurable then the philosophical foundation of the Order would be threatened. It is generally accepted that such a threat did eventuate in Pythagoras' time and that it arose in the investigation of the cherished Pythagorean Theorem.

The discovery which created such a threat to Pythagoreanism was that natural numbers or their ratio are not sufficient when comparing the length of the diagonal of a square to its side. It is not possible to find a Natural number which ***measures*** both the diagonal and the side of a square. Similarly it is not possible to find a small length which can be marked off a whole number of times into both the diagonal and the side. The belief that all the secrets of the Universe would be found in Natural numbers was in tatters. There existed, contrary to intuition, **incommensurable** numbers and line segments. We call the numbers needed to represent the length of the diagonal, the **irrational numbers** meaning - "unable to be expressed as a ratio". The early Pythagoreans have been credited with the discovery of the irrational number $\sqrt{2}$, though it is not clear whether they had expanded their investigations into any other surds. Certainly within the next couple of centuries many of the smaller surds were identified. The discovery of the irrationals and the philosophical difficulties it created in number theory diverted Greek mathematics to the rigours of geometry where the ***measure*** of incommensurables was less troublesome - they could represent a line of length $\sqrt{2}$ even if they couldn't measure it, simply by drawing the diagonal of a unit square. Heath claims:

"...it was inevitable that the Pythagoreans should investigate the relations between sides and hypotenuse of other right-angled triangles. They would naturally give special attention to the isosceles right-angled triangle; they would try to measure the diagonal, would arrive at successive approximations, in rational fractions, to the value of $\sqrt{2}$*, and would find that successive efforts to obtain an exact expression for it failed. It was however an enormous step to conclude that such exact expression was* ***impossible****, and it was this step which the Pythagoreans made. ... The actual method by which the Pythagoreans proved the incommensurability of* $\sqrt{2}$ *with unity, was no doubt that referred to by Aristotle a* ***reductio ad absurdum*** *by which it is proved that, if the diagonal is commensurable with the side, it will follow that the same number is both odd and even."*

There is an activity on this work on the web-site.



The Pythagorean proof is still as elegant and breathtaking in its logical structure today as it was then. It is thought to go something like this:

Suppose AC, the diagonal of a square, to be commensurable with AB, its side.

Let $\alpha:\beta$ *be their ratio with no common factors.*
Then $\alpha > \beta$ *and therefore necessarily > 1.*

Now $AC^2:AB^2 = \alpha^2:\beta^2$
and by Pythagoras $AC^2 = 2AB^2$
so $\alpha^2 = 2\beta^2$

Therefore α^2 *is even and therefore* α *is even.*
Since $\alpha:\beta$ *is in its lowest terms, it follows that* β *must be odd.*

Put $\alpha = 2\gamma$
Then $4\gamma^2 = 2\beta^2$
thus $\beta^2 = 2\gamma^2$
So β^2 *and therefore* β *must be even.*

But β *is also odd, which is impossible.*

Therefore the original supposition that the diagonal of a square is commensurable with its side is not possible. So incommensurable or irrational numbers exist.

If, as has been asserted, the two discoveries of **The Theorem** and the presence of **number patterns in music** started the Pythagoreans along the road to science, surely the discovery of this breathtakingly elegant and beautiful proof must have convinced them that they were on the right road.

In the next Edition of Teaching Mathematics I will describe how the Pythagoreans used geometry to solve algebraic equations up to quadratics.

References:

[1] Kline, Morris B. *Mathematics- The Loss of Certainty*, N.Y.:Oxford University Press, 1980.

[2] Boyer, Carl B. *A History of Mathematics*, N.Y.: John Wiley and Sons, 1968.

[3] Heath, Sir Thomas L., *The Thirteen Books of The Elements Second Edition Vol 1*, N.Y.: Dover Publications (orig 1908).

[4] Koestler, Arthur *The Sleepwalkers - A History of Man's Changing Vision of the Universe*. London: Hutchinson & Co., 1959.

Kleptomaniac: someone who thinks 2 +

‘Double Your Money’
A Resolution to the Paradox

Submitted by Owen Hitchings
St Mary’s College, Woree

In the March 1998 issue of *Teaching Mathematics*, a paradox entitled ‘Double Your Money’ was published with a request for solutions. In the August issue, three responses were published, though the editor felt that none of these really resolved the paradox.

More recently, Owen Hitchings found a paper on the Internet by David Chalmers of the University of California. The paper is titled “The Two-Envelope Paradox: A Complete Analysis?” and can be found at http://ling.ucsc.edu/~chalmers/papers/envelope.html . The gist of the paper follows.

Jake is told that he can choose one of two envelopes, and that one of the envelopes contains twice as much money as the other one. Suppose Jake chooses the left envelope and finds $\$a$. Jake knows that the other envelope contains either $\$\frac{1}{2}a$ or $\$2a$. Jake might realise that the unopened has equal chance of being the envelope of lesser or grater value. Thus, if he swaps, he has 50% chance of ending up with $\$\frac{1}{2}a$ and 50% chance of ending up with $\$2a$. This gives an expected return of $\$1.25a$. This means he is better off swapping.

But of course the paradox arises from the fact that if he had done this calculation before he chose the left envelope, he would have known that the right envelope was the better bet and yet there was nothing to tell him this. In fact, by the same logic, he could have concluded that, had he chosen the right envelope, he would have been better off swapping to the left envelope.

The fallacy arises in the assumption that the other envelope had equal chance of containing $\$\frac{1}{2}a$ or $\$2a$. This will only be the case if the amount of money placed in the lesser envelope is determined by choosing at random from a uniform probability distribution with no upper limit. This cannot be the case without the probability of any finite amount being zero. For any finite amount of money to be at all likely (even possible) to turn up, the probability distribution must taper off.

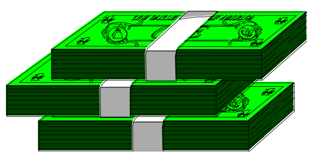
To put this more intuitively, suppose that Tom guessed that the first envelope opened would contain $30 and Dick guessed that it would contain $23 816 048 551 680. Remembering that this is in the context of a quiz show, most people would give Tom a better chance of being exactly correct than Dick. In other words, very large sums are less likely that moderate sums.

Now if the distribution tapers off, then the probability of doubling one’s money by swapping envelopes would be less than the probability of halving it. Therefore the expected return would not necessarily change.

In reality, if Jake had observed that in previous competitions, amounts won averaged $500, then on finding $2000 in his envelope, he would keep it, whereas on finding $40, he would probably ask to swap. Even if he had never seen a previous show, he would have some idea of what amounts of money would be likely to be given away and what amounts would be unlikely.

The paper by David Chalmers gives a full and rigorous (?) mathematical analysis of the situation and is worth a look at. It might be of interest to students of Mathematics B and C as an understandable example of mathematics research.

Many thanks to Owen for this resolution to the problem. Owen will now receive an amount of money which is both twice and half the normal payment for articles submitted to the journal.



The Modern Mathematical Approach to Little Red Riding Hood

A concomitant of the burgeoning of modern mathematics has been the increased use of ingenious symbolism to reduce the writing involved in the communication of mathematical ideas. It seems regrettable that contemporary composers of literature have not seen fit to streamline their communication methods similarly. Through ignorance or design, they remain blissfully aloof of the opportunities afforded to incorporate greater precision as well as an austere beauty into their works. Consider, for instance, how much more satisfying and appealing the following well known classic becomes when clothed with the new notation.

$\exists\, t_0$: at $t = t_0\ \exists$ (note the gain in precision over the time-honoured "Once upon a time there was") a small girl denoted by Little Red Riding Hood (LR^{2}H). LR^{2}H left home, taken as the origin, to pay a visit to $f\,(f\,(\mathrm{LR^2H}))$ (hereinafter denoted G), where f denotes the mapping daughter $\rightarrow$ mother. Her purpose in the visit was the construction of the set $G \cup S$, S being a certain finite subset of the set $C \cup K \cup Q$, where $C = \{\alpha : \alpha = \text{candy}\}$, $K = \{\beta : \beta = \text{cookie}\}$ and $Q = \{\gamma : \gamma = \text{other goodie}\}$.

Now $G \subset F$, a forest and the set $W_1 = \{w : w \text{ alive}, w \in W\}$, where $W = \{w : w = \text{wolf} \in F\}$ was not empty. At $t = t_1 > t_0$, LR^{2}H met $w_1 \in W$, who inquired about the zeros for $t > t1$ of $P'(\mathrm{LR^2H})$, $P(X)$ denoting the position vector of the argument X and the dash denoting the time derivative. On learning that the first zero was at $X = G$, w_1 embarked upon a minimal-variation fixed-end-point path with termini $P(\mathrm{LR^2H})$ and $P(G)$. Arriving at the cottage of G (at $t = t_2$), he effected the transformation $w_1 \cup G \rightarrow w_1$ and, donning the night gear of G, ensconced himself in the latter's bed.

At $t = t_3 > t_2$, the condition $|\,P(\mathrm{LR^2H}) - P(w_1)\,| < \beta$ feet was realised.

Now let $F_1(x)$, $F_2(x)$, $F_3(x)$ denote respectively the ear length, eye brightness and tooth sharpness of x. Let $H_1(y)$, $H2(y)$, $H_3(y)$ denote respectively the efficiency of w_1's acquisition from y of energy of the following types: (1) acoustic, (2) visible electromagnetic and (3) nutritional.

The following conversation then took place, where $E(z)$ denotes, as usual, the expected value of z:

$$\sum_{i=1}^{3} \left[\ \mathrm{LR^2H}\text{: ``}Fi\text{ (you)} >> E(F_i(x)),\ G_0\text{!''};\quad w_1\text{: ``}Fi\text{ (me) maximises } H_i\text{ (you), dear} \in \text{me''}\right]$$

With the last remark for $i = 3$ (at $t = t_4$), w_1 leaped out at LR^{2}H.

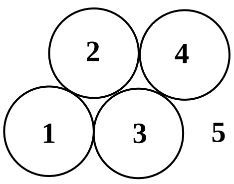
But at $t_5 > t_4 : (t_5 - t_4) \rightarrow 0$, a wood cutter burst into the cottage and axeally transformed w_1 into $w_1 \in W_1$, the complement of W_1 in W. Now, fortunately, $w1$ was separable, with G as one component, and $\exists\ U$ (the universal set) happily $\forall\ t > t_5$.

Source unknown

SOLUTIONS TO EUCLIDEAN COINS

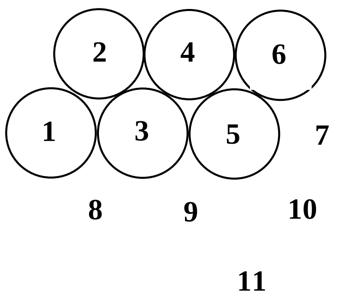
Here are the solutions to two problems posed on page 39 of the last issue of Teaching Mathematics

Problem 1



Place the four coins as shown then slide coin 2 to 5, then coin 3 to 2.

Problem 2



Place the six coins as shown, then slide 3 to 7, 6 to 10, 7 to 11, 10 to 9, 11 to 8.

STUDENT PROBLEM PAGE

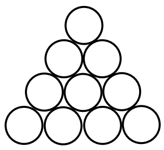
Cheryl Stojanovic Forest Lake College
Garnet Greenbury Publications Committee

Question 1

A store increased the prices of boxed chocolates by 20%. In the clearance sale after Easter, all the prices were reduced by 25%. How do the sale prices compare with the original prices?

Question 2

Boxed chocolate Easter eggs are arranged in a triangular pattern, as shown. The diagram shows 10 eggs arranged with 4 eggs in each of the outside rows. If 78 chocolate eggs were arranged in a similar pattern, how many would there be in the outside rows?

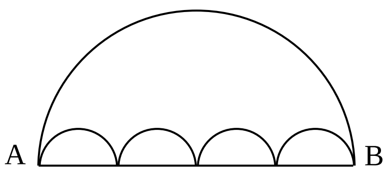


Question 3

At a family reunion, there are 8 teenagers of the same age and five other teenagers of the same age, but different from the first 8. The total of all their ages is 184 years. What age are the older teenagers?

Question 4

Mother Rabbit travels from A to B with one large semi-circular hop. Baby Rabbit travels from A to B using four smaller semi-circular hops as shown in the diagram. Which of the two curved paths is the shortest?



Question 5

Leyah has 100 chocolates. Starting from today, she plans to eat one quarter of the number that remain from the previous day. She will eat chocolates whole and will eat a maximum each day, but never more than a quarter of the remainder. How many chocolates will she eat on day 10 if she follows this plan?

Submitting solutions

Students are invited to submit solutions. Include your name, the problem number, your school and year level (clearly printed). Send them to Garnet Greenbury, Unit 14 Greenleaves Village, Upper Mt Gravatt, 4122. Closing date: 1st July 1999.

Solutions to the problems, Vol. 24 No. 1

1. 10% lower
2. 12 eggs
3. 16 years
4. 2 chocolates
5. Both paths are the same length

Prizes

Prizes for solutions to the student problems in *Teaching Mathematics* Vol. 24 No.1:

Josh Peters of Kingaroy SHS is the winner of the Penguin book prize.

Kath Messer of St Mary's College, Cairns is the winner of the annual subscription to *Tenrag*.

Solutions were received from:

Thulimbah State School
The Gap State High School
Bundaberg State High School
Centenary Heights State High School
St Mary's College, Cairns
Canterbury College
Sienna Catholic College
Stanthorpe State High School
Pimlico State High School
North Bundaberg State High School
Mansfield State High School
Somerville House
Kingaroy State High School
Iona College
Pioneer State High School
Gladstone State High School
Townsville Grammar
Lowood State High School
Emmaus College